

# Higher dimensional analogue of the Blau–Thompson model and $N_T = 8$ , $D = 2$ Hodge–type cohomological gauge theories

B. GEYER<sup>a,b 1</sup> AND D. MÜLSCH<sup>c 2</sup>

<sup>a</sup> *Universität Leipzig, Naturwissenschaftlich-Theoretisches Zentrum  
and Institut für Theoretische Physik, D-04109 Leipzig, Germany*

<sup>b</sup> *Instituto de Fisica, Universidade de Sao Paulo, Sao Paulo, Brasil*

<sup>c</sup> *Wissenschaftszentrum Leipzig e.V., D-04103 Leipzig, Germany*

## Abstract

The higher dimensional analogue of the Blau–Thompson model in  $D = 5$  is constructed by a  $N_T = 1$  topological twist of  $N = 2$ ,  $D = 5$  super Yang–Mills theory. Its dimensional reduction to  $D = 4$  and  $D = 3$  gives rise to the B–model and the  $N_T = 4$  equivariant extension of the Blau–Thompson model, respectively. A further dimensional reduction to  $D = 2$  provides another example of a  $N_T = 8$  Hodge–type cohomological theory with global symmetry group  $SU(2) \otimes \overline{SU(2)}$ . Moreover, it is shown that this theory possesses actually a larger global symmetry group  $SU(4)$  and that it agrees with the  $N_T = 8$  topological twisting of  $N = 16$ ,  $D = 2$  super Yang–Mills theory.

## 1. Introduction

Some very enlightening, but preliminary attempts have been made to incorporate into the gauge–fixing procedure of general gauge theories besides the basic ingredience of the BRST operator  $\Omega$  also a co–BRST operator  $^*\Omega$  which, together with the BRST Laplacian  $W$ , form the same kind of superalgebra as the de Rham cohomology operators in differential geometry [1]. This allows, according to the Hodge–type decomposition  $\psi = \omega + \Omega\chi + ^*\Omega\phi$  of a general quantum state, by imposing both the BRST condition  $\Omega\psi = 0$  and the co–BRST condition  $^*\Omega\psi = 0$  on  $\psi$ , to select the uniquely determined harmonic state  $\omega$  thereby projecting onto the subspace of physical states (for details, see, Section 2 below).

It has been a long–standing problem to present a non–abelian field theoretical model obeying such a Hodge–type cohomological structure. Recently, the authors have shown [2] that the dimensional reduced Blau–Thompson model [3] — the novel  $N_T = 2$  topological twist of the  $N = 4$ ,  $D = 3$  super Yang–Mills theory (SYM) — gives a prototype example of a  $N_T = 4$ ,  $D = 2$  Hodge–type cohomological gauge theory. The conjecture, that topological gauge theories could be possible candidates for Hodge–type cohomological theories was already asserted by van Holten [4]. In fact,  $D = 2$  topological gauge theories [5] are of particular interest because of their relation to  $N = 2$  superconformal theories [6] and Calabi–Yau moduli spaces [7].

In the present paper we construct another example of a 2–dimensional Hodge–type cohomological theory, but now with the largest possible,  $N_T = 8$  topological (co–)shift symmetry and with global symmetry group  $SU(4)$ .<sup>3</sup> This is achieved by first introducing a higher dimensional analogue of the Blau–Thompson model in  $D = 5$  by a  $N_T = 1$  topological twist of  $N = 2$ ,  $D = 5$  SYM with internal symmetry group  $Spin(5) \sim Sp(4)$ . The twisting procedure consists simply in taking the diagonal subgroup of the R–symmetry group  $SO(5)$  and the Euclidean

<sup>1</sup>Email: [geyer@itp.uni-leipzig.de](mailto:geyer@itp.uni-leipzig.de)

<sup>2</sup>Email: [muelsch@informatik.uni-leipzig.de](mailto:muelsch@informatik.uni-leipzig.de)

<sup>3</sup>The existence of such a topological theory was already mentioned in [10], see, footnote on page 248.

rotation group  $SO_E(5)$ . The most unusual feature of this topological model is, analogous to the Blau–Thompson model, that it has no bosonic scalar fields and hence no underlying equivariant cohomology. We conjecture that this new topological theory, which localizes onto the moduli space of complexified flat connections, is the *only* one which can be constructed on a *generic* 5-dimensional Riemannian manifold with  $SO(5)$  holonomy. The other cohomological theories in  $D = 5$ , which localize onto the moduli space of instantons, can be obtained by a dimensional reduction from the higher dimensional analogues of the Donaldson–Witten theory in  $D = 8$  and  $D = 7$  [8, 9, 10]. These theories should be simply untwisted SYM theories formulated on manifolds with reduced holonomy group  $H \subset SO(5)$ .

From this 5-dimensional analogue of the Blau–Thompson model one gets, by an ordinary dimensional reduction to  $D = 4$ , the B-model [11, 3], i.e., one of the 3 inequivalent topological twists of  $N = 4$ ,  $D = 4$  SYM, and by reduction to  $D = 3$  the  $N_T = 4$  *equivariant extension* of the Blau–Thompson model [12]. A further dimensional reduction to  $D = 2$  leads to a Hodge-type cohomological theory with global symmetry group  $SU(2) \otimes \overline{SU}(2)$  and  $N_T = 8$  scalar supercharges. These supercharges, in complete analogy to the de Rham cohomology operators, are interrelated by a discrete Hodge-type  $\star$  operation and generate the topological shift and co-shift symmetries. In accordance with the group theoretical description of some classes of topologically twisted low-dimensional supersymmetric world-volume theories [3], it is shown that this Hodge-type cohomological theory actually allows for the larger global symmetry group  $SU(4)$  [13]. Moreover, it is shown that this theory is precisely the topological twisted  $N = 16$ ,  $D = 2$  SYM with R-symmetry group  $SU(4) \otimes U(1)$ . Such theories are naturally realized as Dirichlet 1-brane instantons wrapping around supersymmetric 2-cycles of Calabi–Yau 2-folds (see, e.g., [14, 15]).

The paper is organized as follows: In Sec. 2 we briefly introduce the BRST complex of general gauge theories and the Hodge-type decomposition. In Sec. 3 we construct the 5-dimensional analogue of the Blau–Thompson model by dimensionally reducing  $N = 1$ ,  $D = 10$  SYM and performing a  $N_T = 1$  topological twist of the resulting Euclidean  $N = 2$ ,  $D = 5$  SYM with R-symmetry group  $SO(5)$ . In Sec. 4 we show that the dimensional reduction of that theory to  $D = 4$  and  $D = 3$  gives rise to the B-model and the  $N_T = 4$  equivariant extension of the Blau–Thompson model, respectively. In Sec. 5 we study the invariance properties of the  $N_T = 8$  Hodge-type cohomological gauge theory with global symmetry group  $SU(2) \otimes \overline{SU}(2)$  obtained by a further dimensional reduction to  $D = 2$ . In Sec. 6 we show that this theory can be cast into a form with the larger global symmetry group  $SU(4)$ . In Sec. 7 we describe in detail the  $N_T = 8$  topological twist of the Euclidean  $N = 16$ ,  $D = 2$  SYM obtained from the  $N = 4$ ,  $D = 4$  SYM via dimensional reduction to  $D = 2$ , and show that it agrees precisely with the Hodge-type cohomological theory with global symmetry group  $SU(4)$ .

## 2. BRST complex and Hodge-type decomposition

In this section we give a rough outlet of the BRST complex, the cohomologies of the (co-) BRST operators and the Hodge-type decomposition as far as it will be used in this paper.

In order to select uniquely the physical states from the ghost-extended quantum state space some attempts [1] have been made to incorporate into the gauge-fixing procedure of general gauge theories besides the BRST operator  $\Omega$  also a co-BRST operator  $\star\Omega$  which, together with the BRST Laplacian  $W$ , obeys the following BRST-complex:

$$\Omega^2 = 0, \quad \star\Omega^2 = 0, \quad W = \{\Omega, \star\Omega\} \neq 0, \quad [\Omega, W] = 0, \quad [\star\Omega, W] = 0, \quad (1)$$

where  $\Omega$  and  $\star\Omega$  have opposite ghost number. Obviously,  $\star\Omega$  can not be identified with the anti-BRST operator  $\bar{\Omega}$  which anticommutes with  $\Omega$ .

Representations of this algebra for the first time have been considered by Nishijima [16]. However, since  $\Omega$  and  $^*\Omega$  are nilpotent hermitian operators they cannot be realized in a Hilbert space. Instead, the BRST complex has to be represented in a Krein space  $\mathcal{K}$  [17].  $\mathcal{K}$  is obtained from a Hilbert space  $\mathcal{H}$  with non-degenerate positive inner product  $(\chi, \psi)$  if  $\mathcal{H}$  will be endowed also with a self-adjoint metric operator  $J \neq 1$ ,  $J^2 = 1$ , allowing for the introduction of another non-degenerate, but indefinite scalar product  $\langle \chi | \psi \rangle := (\chi, J\psi)$ . With respect to the inner product  $\Omega$  and  $^*\Omega = \pm J\Omega J$  are adjoint to each other,  $(\chi, ^*\Omega\psi) = (\Omega\chi, \psi)$ , however they are self-adjoint with respect to the indefinite scalar product of  $\mathcal{K}$ . Notice, that different inner products  $(\chi, \psi)$  lead to different co-BRST operators!

From these definitions one obtains a remarkable correspondence between the BRST cohomology and the de Rham cohomology:

BRST operator	$\Omega$ ,	differential	$d$ ,
co-BRST operator	$^*\Omega = \pm J\Omega J$ ,	co-differential	$\delta = \pm \star d\star$ ,
duality operation	$J$ ,	Hodge star	$\star$ ,
BRST Laplacian	$W = \{\Omega, ^*\Omega\}$ ,	Laplacian	$\Delta = \{d, \delta\}$ .

Because of this correspondence one denotes a state  $\psi$  to be BRST (co-)closed iff  $\Omega\psi = 0$  ( $^*\Omega\psi = 0$ ), BRST (co-)exact iff  $\psi = \Omega\chi$  ( $\psi = ^*\Omega\phi$ ) and BRST harmonic iff  $W\psi = 0$ . Completely analogous to the Hodge decomposition theorem in differential geometry there exists a corresponding decomposition of any state  $\psi$  into a harmonic, an exact and a co-exact state,  $\psi = \omega + \Omega\chi + ^*\Omega\phi$ . The physical properties of  $\psi$  lie entirely within the BRST harmonic part  $\omega$  which is given by the zero modes of  $W$ ; thereby  $W\omega = 0$  implies  $\Omega\omega = 0 = ^*\Omega\omega$ , and vice versa. The cohomologies of the (co-)BRST operator  $\Omega$  (and  $^*\Omega$ ) are given by equivalence classes of states:

$$\begin{aligned} H(\Omega) &= \text{Ker } \Omega / \text{Im } \Omega, & \psi &\sim \psi' = \psi + \Omega\chi, \\ H(^*\Omega) &= \text{Ker } ^*\Omega / \text{Im } ^*\Omega, & \psi &\sim \psi' = \psi + ^*\Omega\phi. \end{aligned}$$

By imposing only the BRST gauge condition,  $\Omega\psi = 0$ , within the equivalence class of BRST-closed states  $\psi = \omega + \Omega\chi$  besides the harmonic state  $\omega$  there occur also spurious BRST-exact states,  $\Omega\chi$ , which have zero physical norm. On the other hand, by imposing also the co-BRST gauge condition,  $^*\Omega\psi = 0$ , one gets for each BRST cohomology class the uniquely determined harmonic state,  $\psi = \omega$ . Obviously, also the observables, being functionals of the fields, are elements of the intersection of the corresponding cohomologies of  $\Omega$  and  $^*\Omega$ .

In the following topological gauge theories are called of Hodge-type if their *scalar* supercharges obey a topological superalgebra quite similar to the BRST complex (1). More precisely, in these theories (which are first-stage reducible) the BRST and co-BRST operators are of the form  $\Omega = Q + s$  and  $^*\Omega = ^*Q + s$ , respectively, where  $Q$  and  $^*Q$  are the generators of the topological shift and co-shift symmetry, and  $s = \delta_G(C)$  is the generator of the ghost-dependent ordinary gauge transformations ( $C$  being the gauge ghosts). Since it is a common practice to ignore in the gauge fermion of topological theories that part which fixes the ordinary gauge symmetry, we always omit in the BRST complex (1) the part stemming from the operator  $s$ . With other words, we look for a topological superalgebra which has precisely the same form as in (1), but with  $\Omega$  and  $^*\Omega$  replaced by  $Q$  and  $^*Q$ , respectively.

### 3. The topological twist of $N = 2$ , $D = 5$ super Yang-Mills theory

In this section we construct a 5-dimensional cohomological gauge theory with a simple,  $N_T = 1$ , scalar supersymmetry  $Q$ . This topological theory has the same interesting feature

as the Blau–Thompson model [3] to possess no bosonic scalar fields and hence no underlying equivariant cohomology. Therefore, it may be considered as a higher dimensional analogue of that model.

In order to get this theory we first dimensionally reduce  $N = 1$ ,  $D = 10$  SYM to  $D = 5$  by breaking down the 10-dimensional Lorentz group according to  $SO(1, 9) \supset SO(1, 4) \otimes Spin(5)$ . The internal symmetry group of this dimensionally reduced  $N = 2$ ,  $D = 5$  SYM is  $Spin(5) \sim Sp(4)$ , the covering of the R-symmetry group  $SO(5)$ . Then, we perform a Wick rotation to Euclidean space and embed the Euclidean rotation group  $SO_E(5)$  into the global symmetry group such that at least one of the supercharges of the untwisted theory becomes a scalar with respect to the new rotation group. This is achieved by taking the diagonal subgroup of  $SO_E(5) \otimes SO(5)$  thereby leading to the  $N_T = 1$  topological twist of the Euclidean  $N = 2$ ,  $D = 5$  SYM. Let us stress that it is necessary to start from the Minkowskian  $N = 1$ ,  $D = 10$  SYM because of the well-known fact that there are no Majorana spinors in Euclidean space. Since the details of this intrinsically 5-dimensional twisting procedure are rather involved we present some of the relevant steps in detail.

First, the Minkowskian action of  $N = 1$ ,  $D = 10$  SYM reads [18],

$$S^{(N=1)} = \int_M d^{10}x \operatorname{tr} \left\{ \frac{1}{4} F^{MN} F_{MN} - i \bar{\lambda} \Gamma^M D_M \lambda \right\}, \quad (2)$$

with  $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$  and  $D_M = \partial_M + [A_M, \cdot]$ . It is build up from an anti-hermitean vector field  $A_M$  ( $M = 0, \dots, 3, 5, \dots, 10$ ) and a Majorana–Weyl spinor  $\lambda$  in the real 16-dimensional representation of  $SO(1, 9)$ . All the fields take their values in the Lie algebra  $Lie(G)$  of some compact gauge group  $G$ . This action is invariant under the following supersymmetry transformations (with 16 real spinorial charges):

$$\begin{aligned} \delta_Q A_M &= \bar{\zeta} \Gamma_M \lambda - \bar{\lambda} \Gamma_M \zeta, \\ \delta_Q \lambda &= \frac{1}{2} i \Gamma^M \Gamma^N \zeta F_{MN}, \\ \delta_Q \bar{\lambda} &= -\frac{1}{2} i \bar{\zeta} \Gamma^M \Gamma^N F_{MN}, \end{aligned} \quad (3)$$

where  $\zeta$  is a constant Majorana–Weyl spinor. This symmetry is checked by using the identity

$$\frac{1}{2} \Gamma^{L_0} [\Gamma^M, \Gamma^N] = \eta^{L_0[M} \Gamma^{N]} - \frac{1}{7!} \epsilon^{L_0 M N L_1 \dots L_7} \Gamma_{11} \Gamma_{L_1} \dots \Gamma_{L_7}, \quad \Gamma_{11} = \Gamma_0 \dots \Gamma_{10},$$

where  $\eta_{MN} = \operatorname{diag}(-1, +1, \dots, +1)$  and  $\epsilon^{L_0 \dots L_{10}}$  are the metric and the completely antisymmetric unit tensor in  $D = 10$ , respectively.

For the 32-dimensional Dirac matrices,  $\{\Gamma_M, \Gamma_N\} = 2\eta_{MN} I_{32}$ , in 10-dimensional Minkowski space-time we choose the following block representation:

$$\begin{aligned} \Gamma_m &= \begin{pmatrix} 0 & (\gamma_m)_a{}^b \delta_A^B \\ (\gamma_m)_a{}^b \delta_A^B & 0 \end{pmatrix}, & m = 0, 1, 2, 3, 5, \\ \Gamma_{5+\alpha} &= i \begin{pmatrix} 0 & \delta_a{}^b (\gamma_\alpha)_A^B \\ -\delta_a{}^b (\gamma_\alpha)_A^B & 0 \end{pmatrix}, & \alpha = 1, 2, 3, 4, 5, \\ \Gamma_{11} &= \begin{pmatrix} \delta_a{}^b \delta_A^B & 0 \\ 0 & -\delta_a{}^b \delta_A^B \end{pmatrix}, & C_{10} = \begin{pmatrix} 0 & (C_5)_{ab} \epsilon_{AB} \\ -(C_5)_{ab} \epsilon_{AB} & 0 \end{pmatrix}. \end{aligned}$$

Here,  $(\gamma_m)_a{}^b$  and  $(\gamma_\alpha)_A^B$  are the  $SO(1, 4)$  and  $SO(5)$  matrices, respectively, where both types of spinor indices  $(a, b)$  and  $(A, B)$  are taking 4 distinct values,

$$\begin{aligned} (\gamma_m)_a{}^b &= (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5), & \gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3, & \gamma_5^2 &= I_4, \\ (\gamma_\alpha)_A^B &= (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5), & \gamma_4 &= -i\gamma_0. \end{aligned}$$

$\gamma_\mu$  ( $\mu = 0, 1, 2, 3$ ) and  $\gamma_5$  are chosen to be equal  $-i$  times the usual  $SO(1, 3)$  Dirac matrices in the Weyl representation with the metric being  $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ . Notice, that the  $SO(5)$  matrices  $(\gamma_\alpha)_A{}^B$  are hermitean.

The real, antisymmetric charge conjugation matrix  $(C_5)_{ab}$  in the 5-dimensional Minkowski space-time has the defining properties

$$(C_5^{-1})^{ac}(\gamma_m)_c{}^d(C_5)_{db} = (\gamma_m)_b{}^a, \quad (C_5)_{ab} = \gamma_5 C_4, \quad C_4 = i\gamma_2\gamma_0,$$

$C_4$  being the usual 4-dimensional charge conjugation matrix. In the 5-dimensional Euclidean space there exists a real, antisymmetric tensor  $\epsilon_{AB}$ , being the invariant tensor of  $Sp(4)$ , which transposes the  $(\gamma_\alpha)_A{}^B$  matrices,

$$\epsilon^{AC}(\gamma_\alpha)_C{}^D\epsilon_{DB} = -(\gamma_\alpha)_B{}^A, \quad \epsilon_{AB} = \gamma_3\gamma_1, \quad \epsilon_{AC}\epsilon^{BC} = \delta_A{}^B,$$

and which can be chosen numerically to be equal to  $(C_5)_{ab}$ . Moreover, this tensor can be used as symplectic metric to raise and lower the spinor index  $A$ , e.g.,  $\epsilon^{AC}(\gamma_\alpha)_C{}^B = (\gamma_\alpha)^{AB}$  and  $(\gamma_\alpha)_A{}^C\epsilon_{CB} = (\gamma_\alpha)_{AB}$ . According to that convention the matrices  $(\gamma_\alpha)_{AB}$  are antisymmetric,  $(\gamma_\alpha)_{AB} = -(\gamma_\alpha)_{BA}$ , and traceless,  $\epsilon^{AB}(\gamma_\alpha)_{AB} = 0$ .

The Weyl condition  $\lambda = \Gamma_{11}\lambda$  and the Majorana condition  $\lambda = C_{10}\bar{\lambda}^T$ ,  $\bar{\lambda} = \lambda^\dagger\Gamma_0$ , restrict a general unconstrained complex 32-spinor in the  $D = 10$  Minkowski space-time to the real 16-spinor  $\lambda$ . The chirality and the symplectic reality condition give rise to the structure

$$\lambda = \begin{pmatrix} \lambda_{aA} \\ 0 \end{pmatrix}, \quad \bar{\lambda} = (0, \bar{\lambda}^{aA}), \quad \lambda_{aA} = (C_5)_{ab}\epsilon_{AB}\bar{\lambda}^{bB},$$

i.e., the 16 surviving spinor components  $\lambda_{aA}$  are constrained by a  $Sp(4)$ -covariant Majorana condition. We further define

$$A_M = (A_m, V_\alpha),$$

where  $V_\alpha$  are the components of the gauge field related to the internal directions  $x^6 \dots x^{10}$ .

As a next step, we compactify 5 of the 10 dimensions by ordinary dimensional reduction, demanding that no field depends on  $x^6, \dots, x^{10}$ . Then, from (2) for the dimensionally reduced action of the Minkowskian  $N = 2$ ,  $D = 5$  SYM with R-symmetry group  $SO(5)$  one obtains

$$S^{(N=2)} = \int_M d^5x \text{tr} \left\{ \frac{1}{4} F^{mn} F_{mn} + \frac{1}{2} D^m V^\alpha D_m V_\alpha + \frac{1}{4} [V^\alpha, V^\beta] [V_\alpha, V_\beta] \right. \\ \left. - i\bar{\lambda}^{aC} (\gamma^m)_a{}^b D_m \lambda_{bC} - \bar{\lambda}^{aC} (\gamma^\alpha)_C{}^D [V_\alpha, \lambda_{aD}] \right\}. \quad (4)$$

In order to get from (4) a cohomological theory we first perform a Wick rotation,  $x_4 = -ix_0$ , into the Euclidean space. Thereby, we relax the symplectic reality conditions and write the Majorana spinors and their conjugated ones just as in the Minkowskian space, but consider these spinor fields, from now on, as *complex*. Hence, hermiticity is abandoned.<sup>4</sup> Afterwards, we twist the Euclidean rotation group  $SO_E(5)$  with the R-symmetry group  $SO(5)$ , or, in other words, we identify the spinor indices  $a$  and  $A$  (as well as  $m$  and  $\alpha$ ). In this way we obtain the twisted action of the Euclidean  $N_T = 1$ ,  $D = 5$  TYM we are looking for,

$$S^{(N_T=1)} = \int_E d^5x \text{tr} \left\{ \frac{1}{4} F^{\alpha\beta} (A + iV) F_{\alpha\beta} (A - iV) + \frac{1}{2} D^\alpha (A) V_\alpha D^\beta (A) V_\beta \right. \\ \left. - i\lambda^{AC} (\gamma^\alpha)_A{}^B D_\alpha (A) \lambda_{BC} - \lambda^{CA} (\gamma^\alpha)_A{}^B [V_\alpha, \lambda_{CB}] \right\}, \quad (5)$$

---

<sup>4</sup>Lost of hermiticity in the Euclidean formulation of a field theory is *not* a problem. The primary reason to impose reality conditions on spinor fields is unitarity, which is needed only in a field theory with *real* time. We are indebted to the referee for pointing out the lack of reality of the cohomological action (7), below.

where now  $V_\alpha$  transforms as a co-vector field of  $A_\alpha$ .

From the transformation rules (3) one gets the twisted supersymmetry transformations

$$\begin{aligned}\delta_Q A_\alpha &= 2\zeta^{AC}(\gamma_\alpha)_A{}^B \lambda_{BC}, \\ \delta_Q V_\alpha &= -2i\zeta^{CA}(\gamma_\alpha)_A{}^B \lambda_{CB}, \\ \delta_Q \lambda_{AB} &= -\frac{1}{2}i(\sigma_{\alpha\beta})_{AC}\zeta_B{}^C F^{\alpha\beta}(A) + (\gamma_\alpha)_A{}^C(\gamma_\beta)_B{}^D \zeta_{CD} D^\alpha(A) V^\beta - \frac{1}{2}i(\sigma_{\alpha\beta})_{BC}\zeta_A{}^C [V^\alpha, V^\beta].\end{aligned}$$

Since there occur no half-integer spin fields in the action (5) we can convert the spinor notation into the more familiar tensor notation by decomposing the twisted spinor fields  $\lambda_{AB}$  as follows,

$$\lambda_{AB} = \frac{1}{2\sqrt{2}} \left\{ (\gamma^\alpha)_{AB} \psi_\alpha + \frac{1}{2}(\sigma^{\alpha\beta})_{AB} \chi_{\alpha\beta} - \epsilon_{AB} \tilde{\eta} \right\}, \quad \lambda^{AB} = \epsilon^{AC} \epsilon^{BD} \lambda_{CD}, \quad (6)$$

where  $\tilde{\eta}$ ,  $\psi_\alpha$  and  $\chi_{\alpha\beta}$  are Grassmann-odd ghost-for-antighost scalar, vector and antisymmetric (traceless) tensor fields, respectively. Here, the 10 generators  $(\sigma_{\alpha\beta})_{AB}$  of the  $Sp(4)$  rotations obey the relations

$$\begin{aligned}(\gamma_\alpha)_A{}^C(\gamma_\beta)_{CB} &= \delta_{\alpha\beta} \epsilon_{AB} - (\sigma_{\alpha\beta})_{AB}, \\ (\gamma_\alpha)_A{}^C(\sigma_{\beta\gamma})_{CB} &= \delta_{\alpha\gamma}(\gamma_\beta)_{AB} - \delta_{\alpha\beta}(\gamma_\gamma)_{AB} - \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta\eta}(\sigma^{\delta\eta})_{AB}, \\ (\sigma_{\alpha\beta})_A{}^C(\sigma_{\gamma\delta})_{CB} &= \delta_{\alpha\gamma}(\sigma_{\beta\delta})_{AB} - \delta_{\alpha\delta}(\sigma_{\beta\gamma})_{AB} + \delta_{\beta\delta}(\sigma_{\alpha\gamma})_{AB} - \delta_{\beta\gamma}(\sigma_{\alpha\delta})_{AB} \\ &\quad + (\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta})\epsilon_{AB} - \epsilon_{\alpha\beta\gamma\delta\eta}(\gamma^\eta)_{AB},\end{aligned}$$

where  $\epsilon_{\alpha\beta\gamma\delta\eta}$  is the completely antisymmetric unit tensor in  $D = 5$ . Notice, that from the first, defining relation it follows that  $(\sigma_{\alpha\beta})_{AB}$ , owing to the antisymmetry of  $(\gamma_\alpha)_{AB}$ , is symmetric,  $(\sigma_{\alpha\beta})_{AB} = (\sigma_{\alpha\beta})_{BA}$ .

Inserting the decomposition (6) into (5) and introducing the bosonic auxiliary field  $Y$ , the virtue of which is to make the topological supersymmetry  $Q$  strictly nilpotent, for the resulting action, with an underlying non-equivariant  $N_T = 1$  topological supersymmetry  $Q$ , we get

$$\begin{aligned}S^{(N_T=1)} &= \int_E d^5x \operatorname{tr} \left\{ \frac{1}{4} F^{\alpha\beta}(A + iV) F_{\alpha\beta}(A - iV) - \frac{1}{4} i \tilde{\chi}^{\alpha\beta\gamma} D_\alpha(A - iV) \chi_{\beta\gamma} \right. \\ &\quad \left. - i \chi^{\alpha\beta} D_\alpha(A + iV) \psi_\beta - i \psi^\alpha D_\alpha(A - iV) \tilde{\eta} - Y D^\alpha(A) V_\alpha - \frac{1}{2} Y^2 \right\}; \quad (7)\end{aligned}$$

$\tilde{\chi}_{\alpha\beta\gamma} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta\eta}\chi^{\delta\eta}$  is the dual of  $\chi_{\alpha\beta}$ . This action can be written as a sum of a topological term ( $Q$ -cocycle) and a  $Q$ -exact term,

$$S^{(N_T=1)} = - \int_E d^5x \operatorname{tr} \left\{ \frac{1}{4} i \tilde{\chi}^{\alpha\beta\gamma} D_\alpha(A - iV) \chi_{\beta\gamma} \right\} + Q\Psi, \quad Q^2 = 0, \quad (8)$$

with the gauge fermion

$$\Psi = \int_E d^5x \operatorname{tr} \left\{ \frac{1}{4} i \chi^{\alpha\beta} F_{\alpha\beta}(A + iV) - \tilde{\eta} D^\alpha(A) V_\alpha - \frac{1}{2} \tilde{\eta} Y \right\}.$$

The topological supersymmetry transformations, generated by  $Q$ , are given by

$$\begin{aligned}Q A_\alpha &= \psi_\alpha, & Q V_\alpha &= -i\psi_\alpha, \\ Q \psi_\alpha &= 0, & Q \chi_{\alpha\beta} &= -i F_{\alpha\beta}(A - iV), \\ Q \tilde{\eta} &= Y, & Q Y &= 0.\end{aligned}$$

This  $N_T = 1$  topological model in  $D = 5$  bears a strong resemblance to the novel  $N_T = 2$  topological twist of  $N = 4$ ,  $D = 3$  SYM constructed by Blau and Thompson [3]. The most

striking feature of this model is that the topological supersymmetry  $Q$  is not equivariantly nilpotent but rather strictly nilpotent, i.e., prior to the introduction of the gauge ghosts  $C$ . Moreover, this model, as promised, has no bosonic scalar fields. Furthermore, the vector field  $V_\alpha$  is combined with  $A_\alpha$  to form the complexified gauge field  $A_\alpha \pm iV_\alpha$ , where  $A_\alpha - iV_\alpha$  is  $Q$ -invariant, which is another remarkable property of that theory.

Since  $Q$  is a  $Sp(4)$ -singlet the twisted action (7) can be put onto a general 5-dimensional Riemannian manifold with Euclidean signature and  $SO(5)$  holonomy group. In this way one gets a topological theory in  $D = 5$  whose key property is the  $Q$ -exactness of its stress tensor, which implies that the correlation functions of  $Q$ -invariant observables are independent of the metric of the manifold.

We conjecture that this higher dimensional analogue of the Blau–Thompson model is the only topological one which can be constructed in  $D = 5$ . In the next Section it will be shown that from (7), after dimensional reduction to  $D = 4$ , one obtains precisely the action of the B-model [11, 3], a certain topological twist of  $N = 4$  SYM, which localizes onto the moduli space of complexified flat connections. However, it was found that there are two more topological twists of  $N = 4$  SYM, namely the A-model [19, 20, 3] and the half-twisted model [20], whose actions localize onto the moduli space of instantons. Thus, one should expect that in  $D = 5$ , apart from the topological model constructed above, there are at least two more cohomological gauge theories as well. But, both these theories are neither topological nor are they twisted versions of  $N = 2$ ,  $D = 5$  SYM. Rather they are untwisted  $N = 2$  SYM theories formulated on manifolds with reduced holonomy group  $H \subset SO(5)$ . Namely, since instantons in  $D = 5$  require the existence of a Hodge self-dual 4-form it must be a singlet with respect to a proper subgroup of  $SO(5)$  and not of  $SO(5)$  itself. Therefore, these theories are only invariant under a certain class of metric variations which preserve the (reduced) holonomy. Perhaps, the simplest way to obtain both these theories is to perform a dimensional reduction to  $D = 5$  of the higher dimensional analogue of the Donaldson–Witten theory in  $D = 8$  and  $D = 7$  on a Joyce manifold with  $Spin(7)$  and  $G_2$  holonomy [8, 9], respectively.

Moreover, it would be interesting to study the relationship between the higher dimensional analogue of the Blau–Thompson model and the dimensionally reduced  $N_T = 2$  and  $N_T = 3$  cohomological gauge theories on a Calabi–Yau 4-fold and on a quaternionic Kähler manifold with  $SU(4)$  and  $Sp(4) \otimes Sp(2)$  holonomy [8, 10], respectively, to  $D = 5$ . However, here we will not further dwell on these issues.

#### 4. The $N_T = 2$ , $D = 4$ topological B-model and the $N_T = 4$ , $D = 3$ equivariant extension of the Blau–Thompson model

We are now going to discuss the relation between the  $N_T = 1$ ,  $D = 5$  topological model constructed above and the  $N_T = 4$  equivariant extension [12] of the Blau–Thompson model in  $D = 3$ . To this end we first perform a  $(4 + 1)$ -decomposition of the action (7), i.e., we split the coordinates into  $x^\alpha = (x^\mu, x^5)$ ,  $\mu = 1, 2, 3, 4$ . Furthermore, we assume that no field depends on  $x^5$ , i.e.,  $\partial_5 = 0$ . As a next step, we rename the fifth component of  $A_\alpha \pm iV_\alpha$ ,  $\chi_{\alpha 5}$  and  $\psi_\alpha$  according to

$$A_5 - iV_5 = G, \quad A_5 + iV_5 = \bar{G}, \quad \chi_{\mu 5} = \tilde{\psi}_\mu, \quad \psi_5 = \eta,$$

reserving the notation  $A_\mu \pm iV_\mu$ ,  $\chi_{\mu\nu}$  and  $\psi_\mu$  for the corresponding fields in  $D = 4$ . Then, after squeezing (7) to  $D = 4$ , we arrive precisely at the action of the topological B-model, with an extended  $N_T = 2$  on-shell equivariantly nilpotent topological supersymmetry, constructed by

Marcus [11],

$$\begin{aligned}
S^{(N_T=2)} = \int_E d^4x \operatorname{tr} \Big\{ & \frac{1}{4} F^{\mu\nu} (A + iV) F_{\mu\nu} (A - iV) + \frac{1}{4} D^\mu (A + iV) G D_\mu (A - iV) \bar{G} \\
& + \frac{1}{4} D^\mu (A - iV) G D_\mu (A + iV) \bar{G} - \frac{1}{8} [G, \bar{G}]^2 - Y D^\mu (A) V_\mu - \frac{1}{2} Y^2 \\
& - i \chi^{\mu\nu} D_\mu (A + iV) \psi_\nu - i \tilde{\chi}^{\mu\nu} D_\mu (A - iV) \tilde{\psi}_\nu + \frac{1}{4} i G \{ \chi^{\mu\nu}, \tilde{\chi}_{\mu\nu} \} \\
& - i \tilde{\psi}^\mu D_\mu (A + iV) \eta - i \psi^\mu D_\mu (A - iV) \tilde{\eta} - i \bar{G} \{ \psi^\mu, \tilde{\psi}_\mu \} + i G \{ \eta, \tilde{\eta} \} \Big\}, \quad (9)
\end{aligned}$$

where  $\tilde{\chi}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \chi^{\rho\sigma}$  is the dual of  $\chi_{\mu\nu}$ . This topological action, which localizes onto the moduli space of the complexified gauge fields  $A_\mu \pm iV_\mu$ , can be regarded also as a deformation of the  $N_T = 1$ ,  $D = 4$  super-BF theory [3]. Notice that in (9) hermiticity is restored.

Obviously, the action (9) is invariant under the following discrete  $Z_2$  symmetry,

$$Z_2 : \quad \begin{bmatrix} A_\mu & V_\mu & G & \bar{G} & Y \\ \chi_{\mu\nu} & \tilde{\chi}_{\mu\nu} & \psi_\mu & \tilde{\psi}_\mu & \eta & \tilde{\eta} \end{bmatrix} \Rightarrow \begin{bmatrix} A_\mu & -V_\mu & G & \bar{G} & -Y \\ \tilde{\chi}_{\mu\nu} & \chi_{\mu\nu} & \tilde{\psi}_\mu & \psi_\mu & \tilde{\eta} & \eta \end{bmatrix},$$

which also maps the topological supercharge  $Q$  into  $\tilde{Q}$ , i.e., the topological supersymmetry is actually  $N_T = 2$ .

Let us now perform a further dimensional reduction of this topological B-model to  $D = 3$ . For that purpose we introduce a  $SU(2)$ - and a  $\overline{SU(2)}$ -doublet of Grassmann-odd ghost-for-antighost scalar and vector fields,  $\eta^a$ ,  $\psi_\alpha^a$  and  $\bar{\eta}^a$ ,  $\bar{\psi}_\alpha^a$ , respectively, and a  $SU(2) \otimes \overline{SU(2)}$ -quartet of Grassmann-even complex scalar fields  $M^{ab}$ , according to

$$\begin{aligned}
\psi_\alpha^a &= \begin{pmatrix} \tilde{\chi}_{4\alpha} \\ \psi_\alpha \end{pmatrix}, \quad \bar{\psi}_\alpha^a = i \begin{pmatrix} \chi_{4\alpha} \\ \tilde{\psi}_\alpha \end{pmatrix}, \quad \eta^a = \begin{pmatrix} \eta \\ \psi_4 \end{pmatrix}, \quad \bar{\eta}^a = i \begin{pmatrix} \tilde{\eta} \\ \tilde{\psi}_4 \end{pmatrix}, \\
M^{ab} &= i \begin{pmatrix} \bar{G} & A_4 - iV_4 \\ A_4 + iV_4 & -G \end{pmatrix}, \quad G = A_5 - iV_5, \quad \bar{G} = A_5 + iV_5, \quad (10)
\end{aligned}$$

where the space index  $\alpha$  runs now between 1 and 3. The internal group index  $a = 1, 2$  is raised and lowered as follows:  $\epsilon^{ac} \varphi_c^b = \varphi^{ab}$  and  $\varphi_a^c \epsilon_{cb} = \varphi_{ab}$ , where  $\epsilon^{ab}$  is the invariant tensor of the group  $SU(2)$ ,  $\epsilon_{12} = \epsilon^{12} = 1$ . The matrices  $M^{ab}$  and  $M_{ab}$  can be rewritten as follows

$$M^{ab} = (\sigma_m)^{ab} M^m, \quad M_{ab} = (\sigma_m^*)^{ab} M^m = M^{cd} \epsilon_{ca} \epsilon_{db}, \quad m = 1, 2, 3, 4,$$

with  $M^m = \{A_{4,5}, V_{4,5}\}$ , where  $(\sigma_m)^{ab} = (i\sigma_1, i\sigma_2, i\sigma_3, -I_2)$  and its hermitean conjugate are the Clebsch-Gordon coefficients relating the vector representation of  $SO(4)$  to the  $(1/2, 1/2)$  representation of  $SU(2) \otimes \overline{SU(2)}$ ,  $\sigma_\alpha$  ( $\alpha = 1, 2, 3$ ) being the Pauli matrices. Moreover, in order to close the topological superalgebra off-shell, we introduce an additional set of bosonic auxiliary fields, namely the complex vector fields  $B_\alpha$  and  $\bar{B}_\alpha$ .

As a result, after that dimensional reduction from the action (9) one gets the  $N_T = 4$ ,  $D = 3$  off-shell equivariant extension of the Blau-Thompson model with global symmetry group  $SU(2) \otimes \overline{SU(2)}$ ,

$$\begin{aligned}
S^{(N_T=4)} = \int_E d^3x \operatorname{tr} \Big\{ & \frac{1}{4} i \epsilon^{\alpha\beta\gamma} B_\gamma F_{\alpha\beta} (A + iV) - \frac{1}{4} i \epsilon^{\alpha\beta\gamma} \bar{B}_\gamma F_{\alpha\beta} (A - iV) - \frac{1}{2} \bar{B}^\alpha B_\alpha \\
& + \frac{1}{2} i \epsilon^{\alpha\beta\gamma} \epsilon_{ab} \psi_\gamma^a D_\alpha (A + iV) \psi_\beta^b - \epsilon_{ab} \eta^a D^\alpha (A + iV) \bar{\psi}_\alpha^b \\
& + \frac{1}{4} D^\alpha (A + iV) M_{ab} D_\alpha (A - iV) M^{ab} - Y D^\alpha (A) V_\alpha - \frac{1}{2} Y^2 \\
& - \frac{1}{2} i \epsilon^{\alpha\beta\gamma} \epsilon_{ab} \bar{\psi}_\gamma^a D_\alpha (A - iV) \bar{\psi}_\beta^b - \epsilon_{ab} \bar{\eta}^a D^\alpha (A - iV) \psi_\alpha^b \\
& + \frac{1}{16} [M_{ab}, M_{cd}] [M^{ab}, M^{cd}] + i M_{ab} \{ \bar{\psi}^{aa}, \psi_\alpha^b \} + i M_{ab} \{ \eta^a, \bar{\eta}^b \} \Big\}. \quad (11)
\end{aligned}$$



The  $N_T = 4$  topological supercharges will be denoted by  $Q^a$  and  $\bar{Q}^a$ . They are interchanged by the following discrete  $Z_2$  symmetry of the action (11),

$$Z_2 : \begin{bmatrix} A_\alpha & V_\alpha & B_\alpha & \bar{B}_\alpha & Y \\ \psi_\alpha^a & \bar{\psi}_\alpha^a & \eta^a & \bar{\eta}^a & M^{ab} \end{bmatrix} \Rightarrow \begin{bmatrix} A_\alpha & -V_\alpha & -\bar{B}_\alpha & -B_\alpha & -Y \\ i\bar{\psi}_\alpha^a & -i\psi_\alpha^a & i\bar{\eta}^a & -i\eta^a & M^{ba} \end{bmatrix}.$$

Thereby,  $Q^a$  and  $\bar{Q}^a$  transform as a doublet of  $SU(2)$  and  $\overline{SU(2)}$ , respectively. The transformation laws of the topological supersymmetry  $Q^a$  are

$$\begin{aligned} Q^a A_\alpha &= \psi_\alpha^a, & Q^a V_\alpha &= -i\psi_\alpha^a, \\ Q^a \psi_\alpha^b &= \epsilon^{ab} B_\alpha, & Q^a \bar{\psi}_\alpha^b &= iD_\alpha(A - iV)M^{ba}, \\ Q^a \eta^b &= 0, & Q^a \bar{\eta}^b &= -i\epsilon^{ab}Y + \frac{1}{2}\epsilon_{cd}[M^{ca}, M^{db}], \\ Q^a M^{bc} &= -2i\epsilon^{ac}\eta^b, & Q^a Y &= -\epsilon_{cd}[M^{ca}, \eta^d], \\ Q^a B_\alpha &= 0, & Q^a \bar{B}_\alpha &= -2D_\alpha(A - iV)\bar{\eta}^a - 2i\epsilon_{cd}[M^{ca}, \bar{\psi}_\alpha^d]. \end{aligned} \quad (12)$$

Combining (12) with the  $Z_2$ -symmetry, which maps  $Q^a$  into  $\bar{Q}^a$ , one gets the corresponding transformation laws of  $\bar{Q}^a$ . It is straightforward to prove that indeed

$$(Q^a, \bar{Q}^a)S^{(N_T=4)} = 0.$$

Furthermore, by an explicit calculation it can be verified that  $Q^a$  and  $\bar{Q}^a$  are both strictly nilpotent and anticommute with each other modulo the field-dependent gauge transformation  $\delta_G(M^{ab})$ , i.e., they satisfy the topological superalgebra off-shell,

$$\{Q^a, Q^b\} = 0, \quad \{\bar{Q}^a, Q^b\} = -2\delta_G(M^{ab}), \quad \{\bar{Q}^a, \bar{Q}^b\} = 0;$$

here, the gauge transformations are defined by  $\delta_G(\varphi)A_\alpha = -D_\alpha\varphi$  and  $\delta_G(\varphi)X = [\varphi, X]$  for all the other fields. This algebraic structure looks like the BRST complex (1). However, both operators,  $Q^a$  and  $\bar{Q}^a$ , are interrelated by the  $Z_2$  symmetry and not, as it should be, by any Hodge-type  $\star$  operation.

Finally, let us mention that precisely the same topological action (11) arises from a dimensional reduction of the A-model [16] to  $D = 3$  [3, 12].

## 5. $N_T = 8$ , $D = 2$ Hodge-type cohomological gauge theory with global symmetry group $SU(2) \otimes \overline{SU(2)}$

Now, we come to the main objective of that paper, namely we show that by a further dimensional reduction of the action (11) to  $D = 2$  we obtain another example of a Hodge-type cohomological gauge theory whose  $N_T = 8$  first-stage reducible gauge symmetries are fixed by harmonic gauges. Indeed, the  $N_T = 4$  topological shift symmetries  $Q^a$ ,  $\bar{Q}^a$  and the  $N_T = 4$  topological co-shift symmetries  $\star Q^a = -P \star Q^a \star$ ,  $\star \bar{Q}^a = -P \star \bar{Q}^a \star$ , being interchanged by a discrete Hodge-type  $\star$  operation, obey the BRST complex (1) (see, Eq. (16) below). Here, and in the following  $P$  denotes the operator of Grassmann-parity whose both eigenvalues  $\pm 1$  are defined through

$$P\varphi = \begin{cases} +\varphi & \text{if } \varphi \text{ is Grassmann-even,} \\ -\varphi & \text{if } \varphi \text{ is Grassmann-odd.} \end{cases}$$

To begin with, we rename the third component of  $A_\alpha \pm iV_\alpha$ ,  $\psi_\alpha^a$  and  $\bar{\psi}_\alpha^a$  according to

$$A_3 - iV_3 = M, \quad A_3 + iV_3 = \bar{M}, \quad \psi_3^a = i\bar{\zeta}^a, \quad \bar{\psi}_3^a = -i\zeta^a. \quad (13)$$

In order to close the topological superalgebra off-shell we introduce an additional set of bosonic auxiliary fields, namely the complex vector fields  $E_\mu$ ,  $\bar{E}_\mu$  and the  $SU(2) \otimes \overline{SU(2)}$ -quartet of complex vector fields  $E_\mu^{ab}$ , where now  $\mu$  denotes the space index taking the values 1 and 2.

Then, after performing in the action (11) the dimensional reduction to  $D = 2$  we arrive at the following  $N_T = 8$  Hodge-type cohomological gauge theory with global symmetry group  $SU(2) \otimes \overline{SU(2)}$ ,

$$\begin{aligned}
S^{(N_T=8)} = \int_E d^2x \operatorname{tr} \{ & \frac{1}{4} i \epsilon^{\mu\nu} B F_{\mu\nu} (A + iV) - \frac{1}{4} i \epsilon^{\mu\nu} \bar{B} F_{\mu\nu} (A - iV) - \frac{1}{2} \bar{B} B \\
& - \epsilon^{\mu\nu} \epsilon_{ab} \bar{\zeta}^a D_\mu (A + iV) \psi_\nu^b + \frac{1}{2} i \epsilon^{\mu\nu} \epsilon_{ab} \bar{M} \{ \psi_\mu^a, \psi_\nu^b \} + i \epsilon_{ab} M \{ \bar{\eta}^a, \bar{\zeta}^b \} \\
& - \epsilon_{ab} \bar{\eta}^a D^\mu (A - iV) \psi_\mu^b + \frac{1}{4} D^\mu (A - iV) M D_\mu (A + iV) \bar{M} \\
& + \frac{1}{4} D^\mu (A + iV) M_{ab} D_\mu (A - iV) M^{ab} - Y D^\mu (A) V_\mu - \frac{1}{2} Y^2 \\
& - \epsilon^{\mu\nu} \epsilon_{ab} \zeta^a D_\mu (A - iV) \bar{\psi}_\nu^b - \frac{1}{2} i \epsilon^{\mu\nu} \epsilon_{ab} M \{ \bar{\psi}_\mu^a, \bar{\psi}_\nu^b \} - i \epsilon_{ab} \bar{M} \{ \eta^a, \zeta^b \} \\
& - \epsilon_{ab} \eta^a D^\mu (A + iV) \bar{\psi}_\mu^b + \frac{1}{4} D^\mu (A + iV) M D_\mu (A - iV) \bar{M} \\
& + \frac{1}{16} [M_{ab}, M_{cd}] [M^{ab}, M^{cd}] + \frac{1}{4} [M, M_{ab}] [\bar{M}, M^{ab}] - \frac{1}{8} [M, \bar{M}]^2 \\
& + i M_{ab} \{ \bar{\psi}^{\mu a}, \psi_\mu^b \} + i M_{ab} \{ \eta^a, \bar{\eta}^b \} + i M_{ab} \{ \zeta^a, \bar{\zeta}^b \} - \bar{E}^\mu E_\mu - \frac{1}{2} E_{ab}^\mu E_\mu^{ab} \}. \quad (14)
\end{aligned}$$

Let us recall that in the  $D = 2$  dimensional Euclidean space there are no propagating degrees of freedom associated with  $A_\mu$ .

The action (14) is manifestly invariant under the following discrete  $Z_2$  symmetry,

$$Z_2 : \begin{bmatrix} A_\mu & V_\mu & & \\ B & \bar{B} & Y & \\ \psi_\mu^a & \eta^a & \zeta^a & M^{ab} \\ \bar{\psi}_\mu^a & \bar{\eta}^a & \bar{\zeta}^a & E_\mu^{ab} \\ M & \bar{M} & E_\mu & \bar{E}_\mu \end{bmatrix} \Rightarrow \begin{bmatrix} A_\mu & -V_\mu & & \\ -\bar{B} & -B & -Y & \\ i\bar{\psi}_\mu^a & i\bar{\eta}^a & i\bar{\zeta}^a & M^{ba} \\ -i\psi_\mu^a & -i\eta^a & -i\zeta^a & -E_\mu^{ba} \\ \bar{M} & M & -\bar{E}_\mu & -E_\mu \end{bmatrix},$$

mapping  $Q^a$  into  $\bar{Q}^a$  as well as  $\star Q^a$  into  $\star \bar{Q}^a$ , which will be defined immediatly.

In addition, the action (14) is also a invariant under the following discrete Hodge-type  $\star$  symmetry, defined by the replacements

$$\varphi \equiv \begin{bmatrix} \partial_\mu & A_\mu & V_\mu & \\ B & \bar{B} & Y & \\ \psi_\mu^a & \eta^a & \zeta^a & M^{ab} \\ \bar{\psi}_\mu^a & \bar{\eta}^a & \bar{\zeta}^a & E_\mu^{ab} \\ M & \bar{M} & E_\mu & \bar{E}_\mu \end{bmatrix} \Rightarrow \star \varphi = \begin{bmatrix} \epsilon_{\mu\nu} \partial^\nu & \epsilon_{\mu\nu} A^\nu & -\epsilon_{\mu\nu} V^\nu & \\ -\bar{B} & -B & -Y & \\ -i\psi_\mu^a & -i\zeta^a & i\eta^a & -M^{ab} \\ -i\bar{\psi}_\mu^a & -i\bar{\zeta}^a & i\bar{\eta}^a & \epsilon_{\mu\nu} E^{\nu ab} \\ -M & -\bar{M} & \epsilon_{\mu\nu} E^\nu & \epsilon_{\mu\nu} \bar{E}^\nu \end{bmatrix},$$

with the property  $\star(\star\varphi) = P\varphi$ , mapping  $Q^a$  and  $\bar{Q}^a$  into  $\star Q^a = -P\star Q^a\star$  and  $\star \bar{Q}^a = -P\star \bar{Q}^a\star$ , respectively.

The topological shift transformations, generated by  $Q^a$ , are given as follows

$$\begin{aligned}
Q^a A_\mu &= \psi_\mu^a, \\
Q^a V_\mu &= -i\psi_\mu^a, \\
Q^a M &= 0, \\
Q^a \bar{\zeta}^b &= -i\epsilon^{ab} B, \\
Q^a B &= 0, \\
Q^a M^{bc} &= -2i\epsilon^{ac} \eta^b, \\
Q^a \eta^b &= 0,
\end{aligned}$$

$$\begin{aligned}
Q^a \psi_\mu^b &= \epsilon^{ab} E_\mu - i \epsilon^{ab} \epsilon_{\mu\nu} D^\nu (A - iV) M, \\
Q^a E_\mu &= 0, \\
Q^a \bar{\psi}_\mu^b &= \epsilon_{\mu\nu} E^{\nu ba} + i D_\mu (A - iV) M^{ba}, \\
Q^a \bar{M} &= 2i \bar{\zeta}^a, \\
Q^a \bar{\eta}^b &= -i \epsilon^{ab} Y + \frac{1}{2} \epsilon^{ab} [M, \bar{M}] + \frac{1}{2} \epsilon_{cd} [M^{ca}, M^{db}], \\
Q^a \zeta^b &= [M^{ba}, M], \\
Q^a Y &= [M, \bar{\zeta}^a] - \epsilon_{cd} [M^{ca}, \eta^d], \\
Q^a \bar{E}_\mu &= \epsilon_{\mu\nu} D^\nu (A + iV) \bar{\zeta}^a + i \epsilon_{\mu\nu} [\bar{M}, \psi^{\nu a}] - D_\mu (A - iV) \bar{\eta}^a - i \epsilon_{cd} [M^{ca}, \bar{\psi}_\mu^d], \\
Q^a \bar{B} &= 2[\bar{\eta}^a, M] - 2\epsilon_{cd} [M^{ca}, \zeta^d], \\
Q^a E_\mu^{bc} &= -\epsilon^{ac} \epsilon_{\mu\nu} D^\nu (A + iV) \eta^b - i \epsilon^{ac} \epsilon_{\mu\nu} [M^b_d, \psi^{\nu d}] - \epsilon^{ac} D_\mu (A - iV) \zeta^b + i \epsilon^{ac} [M, \bar{\psi}_\mu^b]. \quad (15)
\end{aligned}$$

Combining these transformation rules (15) with the  $Z_2$  and/or the Hodge-type  $\star$  symmetry one obtains the complete set of symmetry transformations which leave the action (14) invariant. Indeed, by a rather lengthy calculation one verifies that

$$(Q^a, \bar{Q}^a, \star Q^a, \star \bar{Q}^a) S^{(N_T=8)} = 0.$$

Furthermore, after a straightforward, but tedious calculation one also verifies that  $Q^a$ ,  $\bar{Q}^a$  and  $\star Q^a$ ,  $\star \bar{Q}^a$  provide an off-shell realization of the following topological superalgebra,

$$\begin{aligned}
\{Q^a, Q^b\} &= 0, & \{Q^a, \star Q^b\} &= -2\epsilon^{ab} \delta_G(M), & \{\star Q^a, \star Q^b\} &= 0, \\
\{Q^a, \star \bar{Q}^b\} &= 0, & \{Q^a, \bar{Q}^b\} &= -2\delta_G(M^{ba}), & \{\star Q^a, \bar{Q}^b\} &= 0, \\
\{\bar{Q}^a, \star Q^b\} &= 0, & \{\star \bar{Q}^a, \star Q^b\} &= 2\delta_G(M^{ab}), & \{\star \bar{Q}^a, Q^b\} &= 0, \\
\{\bar{Q}^a, \bar{Q}^b\} &= 0, & \{\star \bar{Q}^a, \bar{Q}^b\} &= 2\epsilon^{ab} \delta_G(\bar{M}), & \{\star \bar{Q}^a, \star \bar{Q}^b\} &= 0. \quad (16)
\end{aligned}$$

Obviously, this superalgebra is analogous to the de Rham cohomology in differential geometry: The nilpotent topological shift and co-shift operators,  $(Q^a, \bar{Q}^a)$  and  $(\star Q^a, \star \bar{Q}^a)$ , correspond to the exterior and the co-exterior derivatives,  $d$  and  $\delta = \pm \star d \star$ , respectively, where both are interrelated by the Hodge-type  $\star$  operations. Furthermore, the various field-dependent gauge generators  $\delta_G(M, \bar{M}, M^{ab})$ , correspond to the Laplacian  $\Delta = \{d, \delta\}$  so that we have indeed a perfect example of a Hodge-type cohomological gauge theory in  $D = 2$ .

## 6. $N_T = 8$ , $D = 2$ Hodge-type cohomological gauge theory with global symmetry group $SU(4)$

The  $Z_2$  symmetry of the action (14) is immediately related to the factorization of the global symmetry group into both the subgroups  $SU(2)$  and  $\overline{SU(2)}$ . Now, we want to show that this action can be cast into a form where it actually has the larger global symmetry group  $SU(4)$ . This is in accordance with the group theoretical description of some classes of topologically twisted low-dimensional supersymmetric world-volume theories given in [3]. Such effective low-energy world-volume theories appear quite naturally in the study of curved D-branes and D-brane instantons wrapping around supersymmetric cocycles for special Lagrangian submanifolds of Calabi-Yau  $n$ -folds [13, 14, 3].

Indeed, the structure of the superalgebra (16) suggests that the 6 scalar fields  $(M, \bar{M}, M^{ab})$  could be combined to form a sextet of the group  $SO(6) \sim SU(4)$  which should be a global

symmetry group of the theory. To elaborate this suggestion we introduce a  $SU(4)$ -quartet of Grassmann-odd vector fields  $\psi_\mu^\alpha$ , two  $SU(4)$ -quartets of Grassmann-odd scalar fields,  $\bar{\eta}_\alpha$  and  $\bar{\zeta}_\alpha$ , which transform as the fundamental and its complex conjugate representation of  $SU(4)$ , respectively, and a  $SU(4)$ -sextet of Grassmann-even complex scalar fields  $M_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}M^{\gamma\delta}$ , which transform as the second-rank complex selfdual representation of  $SU(4)$ ,

$$\begin{aligned}\psi_\mu^\alpha &= \begin{pmatrix} \psi_\mu^a \\ -i\epsilon_{\mu\nu}\bar{\psi}^{\nu a} \end{pmatrix}, & \bar{\eta}_\alpha &= \begin{pmatrix} \bar{\eta}_a \\ i\zeta_a \end{pmatrix}, & \bar{\zeta}_\alpha &= \begin{pmatrix} \bar{\zeta}_a \\ -i\eta_a \end{pmatrix}, \\ M^{\alpha\beta} &= \begin{pmatrix} \epsilon^{ab}M & iM^{ba} \\ -iM^{ab} & \epsilon^{ab}\bar{M} \end{pmatrix}, & M_{\alpha\beta} &= \begin{pmatrix} \epsilon_{ab}\bar{M} & -iM_{ba} \\ iM_{ab} & \epsilon_{ab}M \end{pmatrix},\end{aligned}\quad (17)$$

where from now on  $\alpha$  denotes the internal group index of  $SU(4)$  taking 4 values. By virtue of (10) and (13), it is easily seen that the matrices  $M_{\alpha\beta}$  and  $M^{\alpha\beta}$  can be written in the form

$$M_{\alpha\beta} = (\gamma_m)_{\alpha\beta}M^m, \quad M^{\alpha\beta} = (\gamma_m)^*_{\alpha\beta}M^m = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}M_{\gamma\delta}, \quad m = 1, \dots, 6,$$

with  $M^m = \{A_{3,4,5}, V_{3,4,5}\}$ , where  $(\gamma_m)_{\alpha\beta}$  are the generators of the group  $SO(6)$ ,

$$(\gamma_m)_{\alpha\beta} = (\gamma_1 C_5, \gamma_2 C_5, \gamma_3 C_5, \gamma_4 C_5, \gamma_5 C_5, C_5), \quad \gamma_4 = -i\gamma_0, \quad C_5 = \gamma_5 C_4;$$

the  $SO(1,3)$  Dirac matrices  $\gamma_\mu$  ( $\mu = 0, 1, 2, 3$ ) and  $\gamma_5$  are defined as in Section 3. Furthermore, analogous to  $M_{\alpha\beta}$ , we combine the 6 auxiliary vector fields  $(E_\mu, \bar{E}_\mu, E_\mu^{ab})$  to a  $SU(4)$ -sextet  $E_\mu^{\alpha\beta}$ .

In terms of these new fields from (14) one gets a  $N_T = 8$  Hodge-type cohomological gauge theory with the larger global symmetry group  $SU(4)$ ,

$$\begin{aligned}S^{(N_T=8)} &= \int_E d^2x \text{tr} \left\{ \frac{1}{4}i\epsilon^{\mu\nu}BF_{\mu\nu}(A+iV) - \frac{1}{4}i\epsilon^{\mu\nu}\bar{B}F_{\mu\nu}(A-iV) - \frac{1}{2}\bar{B}B \right. \\ &\quad - \epsilon^{\mu\nu}\bar{\zeta}_\alpha D_\mu(A+iV)\psi_\nu^\alpha - \bar{\eta}_\alpha D^\mu(A-iV)\psi_\mu^\alpha - \frac{1}{4}E_{\alpha\beta}^\mu E_\mu^{\alpha\beta} \\ &\quad + \frac{1}{2}i\epsilon^{\mu\nu}M_{\alpha\beta}\{\psi_\mu^\alpha, \psi_\nu^\beta\} + iM^{\alpha\beta}\{\bar{\eta}_\alpha, \bar{\zeta}_\beta\} - YD^\mu(A)V_\mu - \frac{1}{2}Y^2 \\ &\quad \left. + \frac{1}{8}D^\mu(A+iV)M_{\alpha\beta}D_\mu(A-iV)M^{\alpha\beta} + \frac{1}{64}[M_{\alpha\beta}, M_{\gamma\delta}][M^{\alpha\beta}, M^{\gamma\delta}] \right\}. \quad (18)\end{aligned}$$

In this  $SU(4)$  symmetric form the Hodge-type  $\star$  symmetry is now defined by the replacements

$$\varphi \equiv \begin{bmatrix} \partial_\mu & A_\mu & V_\mu & \\ \psi_\mu^\alpha & \bar{\eta}_\alpha & \bar{\zeta}_\alpha & M^{\alpha\beta} \\ B & \bar{B} & Y & E_\mu^{\alpha\beta} \end{bmatrix} \Rightarrow \star\varphi = \begin{bmatrix} \epsilon_{\mu\nu}\partial^\nu & \epsilon_{\mu\nu}A^\nu & -\epsilon_{\mu\nu}V^\nu & \\ -i\psi_\mu^\alpha & -i\bar{\zeta}_\alpha & i\bar{\eta}_\alpha & -M^{\alpha\beta} \\ -\bar{B} & -B & -Y & \epsilon_{\mu\nu}E^{\nu\alpha\beta} \end{bmatrix}. \quad (19)$$

From (16) one can immediately read off that both the  $SU(2)$ - and  $\overline{SU(2)}$ -doublets  $(Q^a, \bar{Q}^a)$  and  $(\star Q^a, \star \bar{Q}^a)$  should fit into  $SU(4)$ -quartets as follows,

$$Q^\alpha = \begin{pmatrix} Q^a \\ i\star\bar{Q}^a \end{pmatrix}, \quad \star Q^\alpha = -P\star Q^\alpha\star = \begin{pmatrix} \star Q^a \\ i\bar{Q}^a \end{pmatrix},$$

both charges being interrelated by the  $\star$  operation, such that the Hodge-type cohomological superalgebra (16) simplifies into

$$\{Q^\alpha, Q^\beta\} = 0, \quad \{Q^\alpha, \star Q^\beta\} = -2\delta_G(M^{\alpha\beta}), \quad \{\star Q^\alpha, \star Q^\beta\} = 0.$$

Below, in Section 7, it will be shown that the action (18) can be also obtained from the twisted  $N = 16$ ,  $D = 2$  SYM with R-symmetry group  $SU(4) \otimes U(1)$ . Since the relationship between

the twisted and untwisted supercharges is rather involved we shall now describe the complete set of transformations which leave (18) invariant. The transformation rules for the topological shift symmetries  $Q^\alpha$  are

$$\begin{aligned}
Q^\alpha A_\mu &= \psi_\mu^\alpha, \\
Q^\alpha V_\mu &= -i\psi_\mu^\alpha, \\
Q^\alpha M_{\beta\gamma} &= 2i(\delta_\beta^\alpha \bar{\zeta}_\gamma - \delta_\gamma^\alpha \bar{\zeta}_\beta), \\
Q^\alpha \psi_\mu^\beta &= E_\mu^{\alpha\beta} - i\epsilon_{\mu\nu} D^\nu (A - iV) M^{\alpha\beta}, \\
Q^\alpha \bar{\zeta}_\beta &= i\delta_\beta^\alpha B, \\
Q^\alpha B &= 0, \\
Q^\alpha \bar{\eta}_\beta &= i\delta_\beta^\alpha Y + \frac{1}{2}[M^{\alpha\gamma}, M_{\gamma\beta}], \\
Q^\alpha Y &= [M^{\alpha\beta}, \bar{\zeta}_\beta], \\
Q^\alpha \bar{B} &= -2[M^{\alpha\beta}, \bar{\eta}_\beta], \\
Q^\alpha E_{\beta\gamma}^\mu &= \delta_{[\beta}^\alpha (\epsilon^{\mu\nu} D_\nu (A + iV) \bar{\zeta}_{\gamma]} - D^\mu (A - iV) \bar{\eta}_{\gamma]} - i\epsilon^{\mu\nu} [M_{\gamma]\delta}, \psi_\nu^\delta]. \tag{20}
\end{aligned}$$

From combining  $Q^\alpha$  with the above displayed Hodge-type  $\star$  symmetry one gets the corresponding transformation rules for the topological co-shift symmetries  ${}^\star Q^\alpha$ .

Furthermore, by an explicit calculation one can verify that (18) is also invariant under the following *on-shell* vector supersymmetries,

$$\begin{aligned}
\bar{Q}_{\mu\alpha} A_\nu &= \delta_{\mu\nu} \bar{\eta}_\alpha - \epsilon_{\mu\nu} \bar{\zeta}_\alpha, \\
\bar{Q}_{\mu\alpha} V_\nu &= -i\delta_{\mu\nu} \bar{\eta}_\alpha - i\epsilon_{\mu\nu} \bar{\zeta}_\alpha, \\
\bar{Q}_{\mu\alpha} M^{\beta\gamma} &= 2i\epsilon_{\mu\nu} (\delta_\alpha^\beta \psi^{\nu\gamma} - \delta_\alpha^\gamma \psi^{\nu\beta}), \\
\bar{Q}_{\mu\alpha} \bar{\zeta}_\beta &= \epsilon_{\mu\nu} E_{\alpha\beta}^\nu + iD_\mu (A - iV) M_{\alpha\beta}, \\
\bar{Q}_{\mu\alpha} \bar{\eta}_\beta &= E_{\mu\alpha\beta} + i\epsilon_{\mu\nu} D^\nu (A + iV) M_{\alpha\beta}, \\
\bar{Q}_{\mu\alpha} \psi_\nu^\beta &= -2\delta_\alpha^\beta F_{\mu\nu} (A) - 2i\delta_\alpha^\beta D_\mu (A) V_\nu - i\delta_\alpha^\beta \delta_{\mu\nu} Y - i\delta_\alpha^\beta \epsilon_{\mu\nu} \bar{B} + \frac{1}{2}\delta_{\mu\nu} [M_{\alpha\gamma}, M^{\gamma\beta}], \\
\bar{Q}_{\mu\alpha} \bar{B} &= 2i\epsilon_{\mu\nu} D^\nu (A + iV) \bar{\eta}_\alpha, \\
\bar{Q}_{\mu\alpha} Y &= 2iD_\mu (A - iV) \bar{\eta}_\alpha - \epsilon_{\mu\nu} [M_{\alpha\beta}, \psi^{\nu\beta}], \\
\bar{Q}_{\mu\alpha} B &= 2i\epsilon_{\mu\nu} D^\nu (A - iV) \bar{\eta}_\alpha + 4iD_\mu (A) \bar{\zeta}_\alpha + 2[M_{\alpha\beta}, \psi_\mu^\beta], \\
\bar{Q}_{\mu\alpha} E_\nu^{\beta\gamma} &= -\delta_\alpha^{[\beta} D_{[\mu} (A + iV) \psi_{\nu]}^{\gamma]} - \delta_\alpha^{[\beta} \delta_{\mu\nu} D^\rho (A - iV) \psi_\rho^{\gamma]} + i\delta_\alpha^{[\beta} [M^{\gamma]\delta}, \epsilon_{\mu\nu} \bar{\eta}_\delta - \delta_{\mu\nu} \bar{\zeta}_\delta]. \tag{21}
\end{aligned}$$

The vector supercharges  $\bar{Q}_{\mu\alpha}$ , together with  $Q^\alpha$  and  ${}^\star Q^\alpha$ , obey the anticommutation relations

$$\{Q^\alpha, \bar{Q}_{\mu\beta}\} \doteq -2\delta_\beta^\alpha (\partial_\mu + \delta_G(A_\mu - iV_\mu)), \quad \{{}^\star Q^\alpha, \bar{Q}_{\mu\beta}\} \doteq -2\delta_\beta^\alpha (\partial_\mu + \delta_G(A_\mu + iV_\mu)).$$

The action (18) is also invariant under the co-vector supersymmetries

$${}^\star \bar{Q}_{\mu\alpha} = -P \star \bar{Q}_{\mu\alpha} \star \doteq i\bar{Q}_{\mu\alpha},$$

which on-shell, i.e., by using *only* the equations of motion of the auxiliary fields, becomes  $i$  times the vector supersymmetries. Hence, it holds

$$(Q^\alpha, {}^\star Q^\alpha, \bar{Q}_{\mu\alpha}) S^{(N_T=8)} = 0,$$

and the total number of (real) supercharges is actually  $N = 16$ . Let us remark that we were not able to find an appropriate set of auxiliary fields in order to complete the superalgebra of the full set of both scalar *and* vector supercharges off-shell.

Finally, let us mention that both the components of the complexified gauge field,  $A_\alpha \pm iV_\alpha$ , do not possess a harmonic part, although  $A_\alpha - iV_\alpha$  and  $A_\alpha + iV_\alpha$  are, respectively,  ${}^\alpha Q^-$  and  ${}^*Q^\alpha$ -invariant. This is owing to the fact that, instead of the gauge field  $A_\alpha$  and the co-vector field  $V_\alpha$ , in the Euclidean space one can view  $A_\alpha - iV_\alpha$  and  $A_\alpha + iV_\alpha$  as two independent fields belonging to the gauge multiplet of the theory.

## 7. The $N_T = 8$ topological twist of $N = 16$ , $D = 2$ super Yang–Mills theory

Now, as anticipated in the previous section, we show that the  $N_T = 8$  Hodge-type cohomological theory with global symmetry group  $SU(4)$  arises from a topological twist of  $N = 16$ ,  $D = 2$  SYM. A group theoretical description of this twist has been given in [3]: First, one dimensionally reduces  $N = 1, D = 10$  SYM to  $D = 2$  by breaking down the Lorentz group according to  $SO(1, 9) \supset SO(1, 1) \otimes Spin(8)$ , where  $Spin(8)$  is the covering of the R-symmetry group  $SO(8)$  of the dimensionally reduced Minkowskian  $N = 16$ ,  $D = 2$  SYM. Then, one considers the branching  $SO(8) \supset SU(4) \otimes U(1)$  and performs a Wick rotation into the Euclidean space. Afterwards one twists the Euclidean rotation group  $SO_E(2) \sim U_E(1)$  in  $D = 2$  by the  $U(1)$  of the R-symmetry group (by simply adding up the both  $U(1)$  charges), thereby leaving the group  $SU(4)$  intact.

Here, we shall proceed in another way. Starting from Euclidean  $N = 4$ ,  $D = 4$  SYM, which already is manifestly  $SU(4)$ -invariant, and performing a dimensional reduction to  $D = 2$  we get the Euclidean  $N = 16$ ,  $D = 2$  SYM. Then, in order to reveal how this theory should be twisted, we carry out the topological B-twist [11] of the Euclidean  $N = 4$ ,  $D = 4$  SYM as well as a dimensional reduction to  $D = 2$  and compare this twisted theory with the untwisted one and with the topological theory (18). From this comparison one immediately reads off how the Euclidean  $N = 16$ ,  $D = 2$  SYM has to be twisted in order to get the  $N_T = 8$  Hodge-type cohomological theory (18). Since the relationship between the twisted and the untwisted fields is rather complex we describe this twisting procedure in some detail.

The field content of  $N = 4$ ,  $D = 4$  SYM consists of an anti-hermitean gauge field  $A_\mu$ , a Majorana spinor  $\lambda_{A\alpha}$  and its conjugate one  $\bar{\lambda}_{\dot{A}}^\alpha$ , which transform as the fundamental and its complex conjugate representation of  $SU(4)$ , respectively, and a set of complex scalar fields  $G_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}G^{\gamma\delta}$ , which transform as the second-rank complex selfdual representation of  $SU(4)$ . All the fields take their values in the Lie algebra  $Lie(G)$  of the gauge group  $G$ .

In Euclidean space this theory has the invariant action [18]

$$S^{(N=4)} = \int_E d^4x \operatorname{tr} \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \bar{\lambda}_{\dot{A}}^\alpha (\sigma_\mu)^{\dot{A}B} D^\mu \lambda_{B\alpha} + \frac{1}{64} [G_{\alpha\beta}, G_{\gamma\delta}] [G^{\alpha\beta}, G^{\gamma\delta}] \right. \\ \left. - \frac{1}{2} i \lambda_{A\alpha} [G^{\alpha\beta}, \lambda^A_{\dot{\beta}}] - \frac{1}{2} i \bar{\lambda}^{\dot{A}\alpha} [G_{\alpha\beta}, \bar{\lambda}_{\dot{A}}^\beta] + \frac{1}{8} D_\mu G_{\alpha\beta} D^\mu G^{\alpha\beta} \right\}, \quad (22)$$

where the numerically invariant tensors  $(\sigma_\mu)^{\dot{A}B}$  and  $(\sigma_\mu)_{\dot{A}B}$  are the Clebsch–Gordon coefficients relating the  $(1/2, 1/2)$  representation of  $SL(2, C)$  to the vector representation of  $SO(4)$ ,

$$(\sigma_\mu)^{\dot{A}B} = (-i\sigma_1, -i\sigma_2, -i\sigma_3, I_2), \quad (\sigma_\mu)_{\dot{A}B} \equiv (\sigma_\mu)^{\dot{C}D} \epsilon_{\dot{C}\dot{A}} \epsilon_{DB} = (\sigma_\mu^*)^{\dot{A}B}, \\ (\sigma_\mu)_{\dot{A}\dot{B}} = (i\sigma_1, i\sigma_2, i\sigma_3, I_2), \quad (\sigma_\mu)^{AB} \equiv \epsilon^{AC} \epsilon^{\dot{B}\dot{D}} (\sigma_\mu)_{C\dot{D}} = (\sigma_\mu^*)_{\dot{A}\dot{B}}, \quad (23)$$

$(\sigma_\mu)_{\dot{A}B}$  and  $(\sigma_\mu)^{AB}$  being the corresponding complex conjugate coefficients. The selfdual and

anti-selfdual generators of the  $SO(4)$  rotations,  $(\sigma_{\mu\nu})_{AB}$  and  $(\sigma_{\mu\nu})_{\dot{A}\dot{B}}$ , obey the relations

$$\begin{aligned}(\sigma_\mu)^{AC}(\sigma_\nu)_{\dot{C}}{}^B &= (\sigma_{\mu\nu})^{AB} - \delta_{\mu\nu}\epsilon^{AB}, \\(\sigma_\rho)^{AC}(\sigma_{\mu\nu})_{\dot{C}}{}^{\dot{B}} &= \delta_{\rho\mu}(\sigma_\nu)^{A\dot{B}} - \delta_{\rho\nu}(\sigma_\mu)^{A\dot{B}} - \epsilon_{\mu\nu\rho\sigma}(\sigma^\sigma)^{A\dot{B}},\end{aligned}\tag{24}$$

$$\begin{aligned}(\sigma_\mu)_{\dot{A}C}(\sigma_\nu)_{\dot{B}}{}^C &= (\sigma_{\mu\nu})_{\dot{A}\dot{B}} + \delta_{\mu\nu}\epsilon_{\dot{A}\dot{B}}, \\(\sigma_\rho)_{\dot{A}C}(\sigma_{\mu\nu})_{\dot{B}}{}^C &= \delta_{\rho\mu}(\sigma_\nu)_{\dot{A}\dot{B}} - \delta_{\rho\nu}(\sigma_\mu)_{\dot{A}\dot{B}} + \epsilon_{\mu\nu\rho\sigma}(\sigma^\sigma)_{\dot{A}\dot{B}}.\end{aligned}\tag{25}$$

The spinor index  $A$  (and analogous  $\dot{A}$ ) is raised and lowered as follows:  $\epsilon^{AC}\varphi_C^B = \varphi^{AB}$  and  $\varphi_A^C\epsilon_{CB} = \varphi_{AB}$ , where  $\epsilon_{AB}$  (and analogous  $\epsilon_{\dot{A}\dot{B}}$ ) is the invariant tensor of the group  $SU(2)$ ,  $\epsilon_{12} = \epsilon^{12} = \epsilon_{\dot{1}\dot{2}} = \epsilon^{\dot{1}\dot{2}} = 1$ .

The action (22) is manifestly invariant under hermitean conjugation:

$$(A_\mu, \lambda_{A\alpha}, \bar{\lambda}^{\dot{A}\alpha}, G^{\alpha\beta}) \rightarrow (-A_\mu, \bar{\lambda}_{\dot{A}}{}^\alpha, \lambda^A{}_\alpha, G_{\alpha\beta}).$$

Furthermore, making use of (24) and (25), by a brief calculation one verifies that (22) is invariant under the following *on-shell* supersymmetry transformations,

$$\begin{aligned}Q_A{}^\alpha A_\mu &= -i(\sigma_\mu)_{A\dot{B}}\bar{\lambda}^{\dot{B}\alpha}, \\Q_A{}^\alpha \bar{\lambda}_{\dot{B}}{}^\beta &= (\sigma^\mu)_{A\dot{B}}D_\mu G^{\alpha\beta}, \\Q_A{}^\alpha G_{\beta\gamma} &= 2i(\delta_\beta^\alpha \lambda_{A\gamma} - \delta_\gamma^\alpha \lambda_{A\beta}), \\Q_A{}^\alpha \lambda_{B\beta} &= -\frac{1}{2}\delta_\beta^\alpha (\sigma^{\mu\nu})_{AB}F_{\mu\nu} - \frac{1}{2}\epsilon_{AB}[G^{\alpha\gamma}, G_{\gamma\beta}]\end{aligned}$$

and

$$\begin{aligned}\bar{Q}_{\dot{A}\alpha} A_\mu &= i(\sigma_\mu)_{\dot{A}B}\lambda^B{}_\alpha, \\\bar{Q}_{\dot{A}\alpha} \lambda_{B\beta} &= (\sigma^\mu)_{\dot{A}B}D_\mu G_{\alpha\beta}, \\\bar{Q}_{\dot{A}\alpha} G^{\beta\gamma} &= 2i(\delta_\alpha^\beta \bar{\lambda}_{\dot{A}}{}^\gamma - \delta_\alpha^\gamma \bar{\lambda}_{\dot{A}}{}^\beta), \\\bar{Q}_{\dot{A}\alpha} \bar{\lambda}_{\dot{B}}{}^\beta &= -\frac{1}{2}\delta_\alpha^\beta (\sigma^{\mu\nu})_{\dot{A}\dot{B}}F_{\mu\nu} + \frac{1}{2}\epsilon_{\dot{A}\dot{B}}[G_{\alpha\gamma}, G^{\gamma\beta}].\end{aligned}$$

Let us recall that it is not possible to complete the superalgebra off-shell with a finite number of auxiliary fields [21].

In order to perform in (22) a dimensional reduction to  $D = 2$  we rename the third and fourth component of  $A_\mu$  according to

$$A_3 = \frac{1}{2}(\phi + \bar{\phi}), \quad A_4 = \frac{1}{2}i(\phi - \bar{\phi}),\tag{26}$$

reserving the notation  $A_\mu$  ( $\mu = 1, 2$ ) for the gauge field in  $D = 2$ . Moreover, we decompose the components of  $(\sigma_\mu)_{\dot{A}}{}^B$ ,  $(\sigma_{\mu\nu})_{\dot{A}}{}^{\dot{B}}$  and  $(\sigma_\mu)_A{}^{\dot{B}}$ ,  $(\sigma_{\mu\nu})_A{}^B$  in the following manner,

$$\begin{aligned}(\sigma_\mu)_{\dot{A}}{}^B &\rightarrow i(\sigma_\mu)_A{}^B, & (\sigma_\mu)_A{}^{\dot{B}} &\rightarrow i(\sigma_\mu)_A{}^B, \\(\sigma_3)_{\dot{A}}{}^B &\rightarrow -i(\sigma_3)_A{}^B, & (\sigma_3)_A{}^{\dot{B}} &\rightarrow -i(\sigma_3)_A{}^B, \\(\sigma_4)_{\dot{A}}{}^B &\rightarrow \delta_A{}^B, & (\sigma_4)_A{}^{\dot{B}} &\rightarrow -\delta_A{}^B, \\(\sigma_{\mu\nu})_{\dot{A}}{}^{\dot{B}} &\rightarrow -i\epsilon_{\mu\nu}(\sigma_3)_A{}^B, & (\sigma_{\mu\nu})_A{}^B &\rightarrow i\epsilon_{\mu\nu}(\sigma_3)_A{}^B, \\(\sigma_{\mu 3})_{\dot{A}}{}^{\dot{B}} &\rightarrow i\epsilon_{\mu\nu}(\sigma^\nu)_A{}^B, & (\sigma_{\mu 3})_A{}^B &\rightarrow -i\epsilon_{\mu\nu}(\sigma^\nu)_A{}^B, \\(\sigma_{\mu 4})_{\dot{A}}{}^{\dot{B}} &\rightarrow i(\sigma_\mu)_A{}^B, & (\sigma_{\mu 4})_A{}^B &\rightarrow i(\sigma_\mu)_A{}^B, \\(\sigma_{34})_{\dot{A}}{}^{\dot{B}} &\rightarrow -i(\sigma_3)_A{}^B, & (\sigma_{34})_A{}^B &\rightarrow -i(\sigma_3)_A{}^B,\end{aligned}\tag{27}$$

such that both the relations (24) and (25) become the algebra of the Pauli matrices (observe that  $(\sigma_\mu, \sigma_3)_A^B \equiv (\sigma_1, \sigma_2, \sigma_3)$ ),

$$\begin{aligned}(\sigma_\mu)_A^C (\sigma_\nu)_{CB} &= \delta_{\mu\nu} \epsilon_{AB} + i \epsilon_{\mu\nu} (\sigma_3)_{AB}, \\ (\sigma_\mu)_A^C (\sigma_3)_{CB} &= -i \epsilon_{\mu\nu} (\sigma^\nu)_{AB}, \\ (\sigma_3)_A^C (\sigma_3)_{CB} &= \epsilon_{AB}.\end{aligned}$$

Then, from the action (22), for the Euclidean action of the  $N = 16$ ,  $D = 2$  SYM we obtain

$$\begin{aligned}S^{(N=16)} = \int_E d^2x \operatorname{tr} \bigg\{ & \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \bar{\phi} D^\mu \phi - \frac{1}{8} [\bar{\phi}, \phi]^2 \\ & - \frac{1}{2} \bar{\lambda}_A^\alpha (\sigma_3)^{AB} [\phi + \bar{\phi}, \lambda_{B\alpha}] + \frac{1}{2} \bar{\lambda}^{A\alpha} [\phi - \bar{\phi}, \lambda_{A\alpha}] \\ & + \bar{\lambda}_A^\alpha (\sigma_\mu)^{AB} D^\mu \lambda_{B\alpha} - \frac{1}{2} i \lambda_{A\alpha} [G^{\alpha\beta}, \lambda^A_{\beta}] - \frac{1}{2} i \bar{\lambda}^{A\alpha} [G_{\alpha\beta}, \bar{\lambda}_A^\beta] \\ & + \frac{1}{8} D_\mu G_{\alpha\beta} D^\mu G^{\alpha\beta} + \frac{1}{8} [\bar{\phi}, G_{\alpha\beta}] [\phi, G^{\alpha\beta}] + \frac{1}{64} [G_{\alpha\beta}, G_{\gamma\delta}] [G^{\alpha\beta}, G^{\gamma\delta}] \bigg\}. \quad (28)\end{aligned}$$

Since the decompositions (27) include some factors of  $i$ , the action (28) is no longer manifestly invariant under hermitean conjugation. Rather, it is invariant under the following  $Z_2$  symmetry,

$$Z_2 : \quad (A_\mu, \phi, \bar{\phi}, \lambda_{A\alpha}, \bar{\lambda}^{A\alpha}, G^{\alpha\beta}) \rightarrow (A_\mu, \bar{\phi}, \phi, -\bar{\lambda}_A^\alpha, -\lambda^A_{\alpha}, G_{\alpha\beta}). \quad (29)$$

If we denote the  $N = 16$  spinorial supercharges in  $D = 2$  with  $Q_A^\alpha$  and  $\bar{Q}_{A\alpha}$ , which are interchanged by the  $Z_2$  symmetry (29), the transformation rules for  $Q_A^\alpha$  are

$$\begin{aligned}Q_A^\alpha A_\mu &= (\sigma_\mu)_{AB} \bar{\lambda}^{B\alpha}, \\ Q_A^\alpha \phi &= -(\sigma_3)_{AB} \bar{\lambda}^{B\alpha} - \bar{\lambda}_A^\alpha, \\ Q_A^\alpha \bar{\phi} &= -(\sigma_3)_{AB} \bar{\lambda}^{B\alpha} + \bar{\lambda}_A^\alpha, \\ Q_A^\alpha G_{\beta\gamma} &= 2i(\delta_\beta^\alpha \lambda_{A\gamma} - \delta_\gamma^\alpha \lambda_{A\beta}), \\ Q_A^\alpha \bar{\lambda}_B^\beta &= \frac{1}{2} i (\sigma^\mu)_{AB} D_\mu G^{\alpha\beta} - \frac{1}{2} i (\sigma_3)_{AB} [\phi + \bar{\phi}, G^{\alpha\beta}] - \frac{1}{2} i \epsilon_{AB} [\phi - \bar{\phi}, G^{\alpha\beta}], \\ Q_A^\alpha \lambda_{B\beta} &= \frac{1}{2} i \delta_\beta^\alpha \epsilon^{\mu\nu} (\sigma_\nu)_{AB} D_\mu (\phi + \bar{\phi}) + \frac{1}{2} \delta_\beta^\alpha (\sigma^\mu)_{AB} D_\mu (\phi - \bar{\phi}) \\ &\quad + \frac{1}{2} \delta_\beta^\alpha (\sigma^3)_{AB} [\phi, \bar{\phi}] - \frac{1}{2} i \delta_\beta^\alpha \epsilon^{\mu\nu} (\sigma_3)_{AB} F_{\mu\nu} - \frac{1}{2} \epsilon_{AB} [G^{\alpha\gamma}, G_{\gamma\beta}]. \quad (30)\end{aligned}$$

Let us now derive the relationship between the action (28) and the cohomological theory (18). Its complexity stems from a nontrivial mixing of the scalar and vector supercharges  $Q^\alpha$ ,  $\star Q^\alpha$ ,  $\bar{Q}_{\mu\alpha}$  and the spinorial supercharges  $Q^{A\alpha}$  and  $\bar{Q}_{A\alpha}$ , namely, the internal indices  $\alpha$  in both theories cannot be simply identified with each other. For that reason, as explained above, we proceed as follows: First of all, we perform in (22) a topological B-twist [11], i.e., we break the group  $SU(4)$  down to  $SU(2) \otimes \overline{SU(2)} \otimes U(1)$  and identify the components  $\alpha = (1, 2)$  and  $\alpha = (3, 4)$  with the spinor indices  $B = (1, 2)$  and  $\dot{B} = (\dot{1}, \dot{2})$  of both subgroups of the Euclidean rotation group  $SO(4) = SU(2)_L \otimes SU(2)_R$ , respectively. Then, we replace in (22) the fields  $\lambda_{A\alpha}$ ,  $\bar{\lambda}^{A\alpha}$  and  $G_{\alpha\beta}$  by the twisted fields  $\eta$ ,  $\tilde{\eta}$ ,  $\psi_\mu$ ,  $\tilde{\psi}_\mu$ ,  $\chi_{\mu\nu}$  and  $G$ ,  $\bar{G}$ ,  $V_\mu$  of the B-model according to

$$\begin{aligned}\lambda_{A\alpha} &= \frac{1}{2} \begin{pmatrix} \epsilon_{AB} (\eta + \tilde{\eta}) - \frac{1}{4} (\sigma^{\mu\nu})_{AB} (\chi_{\mu\nu} + \tilde{\chi}_{\mu\nu}) \\ (\sigma^\mu)_{A\dot{B}} (\psi_\mu - \tilde{\psi}_\mu) \end{pmatrix}, \\ \bar{\lambda}^{A\alpha} &= \frac{1}{2} \begin{pmatrix} -(\sigma^\mu)^{\dot{A}B} (\tilde{\psi}_\mu + \psi_\mu) \\ \epsilon^{\dot{A}\dot{B}} (\tilde{\eta} - \eta) - \frac{1}{4} (\sigma^{\mu\nu})^{\dot{A}\dot{B}} (\tilde{\chi}_{\mu\nu} - \chi_{\mu\nu}) \end{pmatrix}, \quad (31)\end{aligned}$$



and

$$\begin{aligned} G_{\alpha\beta} &= \begin{pmatrix} \epsilon_{AB}\bar{G} & i(\sigma^\mu)_{A\dot{B}}V_\mu \\ -i(\sigma^\mu)_{\dot{A}B}V_\mu & \epsilon_{\dot{A}\dot{B}}G \end{pmatrix}, \\ G^{\alpha\beta} &= \begin{pmatrix} \epsilon^{AB}G & -i(\sigma^\mu)^{A\dot{B}}V_\mu \\ i(\sigma^\mu)^{\dot{A}B}V_\mu & \epsilon^{\dot{A}\dot{B}}\bar{G} \end{pmatrix}. \end{aligned} \quad (32)$$

By using the explicit form (23) of the Clebsch–Gordon coefficients one establishes that  $G_{\alpha\beta}$  and  $G^{\alpha\beta}$  in (32) are actually dual to each other,  $G_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}G^{\gamma\delta}$ . In that way, by making use of the relations (24) and (25), one gets precisely the action (9) of the B-model.

As a next step, we perform in (31) and (32) the decompositions (27) and compare directly the resulting twisted action with (18) — after the elimination of the auxiliary fields  $B$ ,  $\bar{B}$ ,  $E_\mu^{\alpha\beta}$  and  $Y$  through their equations of motion. From this comparison one can deduce that, indeed, there is a unique relationship between each component of the twisted fields which enter into (31) and (32), and the complex scalar fields  $\phi$  and  $\bar{\phi}$  introduced in (26), as well as the whole set of fields  $\psi_\mu^\alpha$ ,  $\bar{\eta}_\alpha$ ,  $\bar{\zeta}_\alpha$  and  $M_{\alpha\beta}$  which enter into (18).

Collecting these results, after a lengthly calculation one obtains the following relationships:

$$\begin{aligned} \lambda_{A\alpha} &= \frac{1}{2} \left( i(\sigma^\mu)_{AB}(\psi_\mu^1 - \epsilon_{\mu\nu}\psi^{\nu 3}) + (\sigma_3)_{AB}(\bar{\eta}_4 + \bar{\zeta}_2) + i\epsilon_{AB}(\bar{\zeta}_4 - \bar{\eta}_2) \right), \\ \bar{\lambda}^{A\alpha} &= \frac{1}{2} \left( i(\sigma^\mu)^{AB}(\epsilon_{\mu\nu}\psi^{\nu 4} - \psi_\mu^2) + (\sigma_3)^{AB}(\bar{\zeta}_1 + \bar{\eta}_3) - i\epsilon^{AB}(\bar{\eta}_1 - \bar{\zeta}_3) \right), \end{aligned} \quad (33)$$

between  $\lambda_{A\alpha}$ ,  $\bar{\lambda}^{A\alpha}$  and the twisted spinor fields  $\psi_\mu^\alpha$ ,  $\bar{\eta}_\alpha$ ,  $\bar{\zeta}_\alpha$ , and

$$\begin{aligned} \phi &= A_3 - iA_4, & \bar{\phi} &= A_3 + iA_4, \\ G_{\alpha\beta} &= \begin{pmatrix} \epsilon_{AB}\bar{G} & -(\sigma^\mu)_{AB}V_\mu + (\sigma_3)_{AB}V_3 - i\epsilon_{AB}V_4 \\ (\sigma^\mu)_{AB}V_\mu - (\sigma_3)_{AB}V_3 - i\epsilon_{AB}V_4 & \epsilon_{AB}G \end{pmatrix}, \\ G^{\alpha\beta} &= \begin{pmatrix} \epsilon^{AB}G & (\sigma^\mu)^{AB}V_\mu - (\sigma_3)^{AB}V_3 + i\epsilon^{AB}V_4 \\ -(\sigma^\mu)^{AB}V_\mu + (\sigma_3)^{AB}V_3 + i\epsilon^{AB}V_4 & \epsilon^{AB}\bar{G} \end{pmatrix}, \end{aligned} \quad (34)$$

where

$$\begin{aligned} A_3 &= \frac{1}{2}(M^{12} + M^{34}), & V_3 &= \frac{1}{2}i(M^{12} - M^{34}), & G &= M^{24}, \\ A_4 &= \frac{1}{2}(M^{14} + M^{23}), & V_4 &= \frac{1}{2}i(M^{14} - M^{23}), & \bar{G} &= M^{31}, \end{aligned} \quad (35)$$

between  $\phi$ ,  $\bar{\phi}$ ,  $G_{\alpha\beta}$  and the twisted vector and scalar fields  $V_\mu$  and  $M_{\alpha\beta}$ , respectively.

Thereby, the assignment between the spinor indices  $(A, B)$  and the group indices  $(\alpha, \beta)$  is similar as before, e.g., in (34) the spinor indices  $A = 1, 2$  (resp.  $B = 1, 2$ ) at the upper and lower row (resp. at the left and right column) of the both matrices correspond to the values  $\alpha = 1, 2$  and  $\alpha = 3, 4$  (resp.  $\beta = 1, 2$  and  $\beta = 3, 4$ ) of the scalar fields  $G_{\alpha\beta}$ , respectively. Let us also notice, that the relationships (35) agree precisely with the definition (17) of the matrix  $M^{\alpha\beta}$  (c.f., Eqs. (10) and (13)).

As an additional check, inserting (33) – (35) into (28), after a tedious calculation one verifies that the resulting twisted action actually agrees with the topological action (18).

The relationship between the spinorial supercharges  $Q^{A\alpha}$  and  $\bar{Q}_{A\alpha}$ , being interrelated by the replacements (29), and the twisted scalar and vector supercharges  $Q^\alpha$ ,  $\bar{Q}_{\mu\alpha}$  and  ${}^\star Q^\alpha$ ,  ${}^\star \bar{Q}_{\mu\alpha}$ , being interchanged by the  $\star$  operation (19), is quite similar to the ones of the spinor fields, Eq.

(33), namely

$$Q^{A\alpha} = \frac{1}{2} \left( i(\sigma^\mu)^{AB} (\bar{Q}_{\mu 1} - \epsilon_{\mu\nu} \bar{Q}_3^\nu) - (\sigma_3)^{AB} (Q^4 - i^* Q^2) - \epsilon^{AB} (^* Q^4 - i Q^2) \right),$$

$$\bar{Q}_{A\alpha} = \frac{1}{2} \left( i(\sigma^\mu)_{AB} (\epsilon_{\mu\nu} \bar{Q}_4^\nu - \bar{Q}_{\mu 2}) + (\sigma_3)_{AB} (i^* Q^1 - Q^3) + \epsilon_{AB} (i Q^1 - ^* Q^3) \right),$$

$$\bar{Q}_{A\alpha} = \frac{1}{2} \left( i(\sigma^\mu)_{AB} (\epsilon_{\mu\nu} \bar{Q}_3^\nu + \bar{Q}_{\mu 1}) + (\sigma_3)_{AB} (i^* Q^2 + Q^4) + \epsilon_{AB} (i Q^2 + ^* Q^4) \right),$$

i.e., after the twisting the  $Z_2$  symmetry (29) changes into the Hodge-type  $\star$  symmetry (19).

Finally, let us mention that there is also a  $N_T = 4$  topological twist of  $N = 16$ ,  $D = 2$  SYM with global symmetry group  $SO(4) \otimes SU(2)$ . This topological theory can be regarded as the  $N_T = 4$  super-BF theory coupled to a spinorial hypermultiplet. Another way of obtaining the action of this theory is to dimensionally reduce either the higher dimensional analogue of the Donaldson–Witten theory in  $D = 8$  [8, 9] to  $D = 2$  or to dimensionally reduce the  $N_T = 1$  half-twisted theory [17] in  $D = 4$  to  $D = 2$ . However, that topological twist does not lead to another Hodge-type cohomological theory, since the underlying cohomology is equivariantly nilpotent and not strictly nilpotent as it should be.

## Concluding remarks

In this paper we have given a further example of a Hodge-type cohomological gauge theory in  $D = 2$ , but this time with maximal number of  $N_T = 8$  topological supercharges and largest global symmetry group  $SU(4)$ . This topological theory can be obtained either by a  $N_T = 1$  topological twist of the  $N = 2$ ,  $D = 5$  SYM with R-symmetry group  $SO(5)$  and performing afterwards a ordinary dimensional reduction to  $D = 2$  or by a  $N_T = 8$  topological twist of the  $N = 16$ ,  $D = 2$  SYM with R-symmetry group  $SU(4) \otimes U(1)$ . This example gives rise to the conjecture that, in general, the gauge-fixing procedure based on harmonic gauges seems to be possible only for very constrained gauge systems with  $N_T \geq 4$  scalar supercharges.

So far, within our model we have taken into account only the equivariant part of the BRST complex, i.e., the part associated with the shift and co-shift symmetries  $Q^\alpha$  and  $^* Q^\alpha$ . But, it should be emphasized that the  $Q^\alpha$ - and  $^* Q^\alpha$ -cohomology, although empty in the space of unrestricted local functionals of the fields, becomes nonempty if gauge invariance is imposed on these functionals [22]. Therefore, it is important to prove whether or not for the basic cohomology, i.e., the BRST complex including also the ordinary gauge symmetry  $\delta_G(C)$ , the underlying Hodge-structure of the theory can be preserved as well. This, of course, is a nontrivial task and will be the subject of a subsequent paper.

In that paper, we have only found examples in  $D = 2$ , where the gauge fields have no local dynamics and where the Hodge-type  $\star$  symmetry can be rather simply implemented into the theory. In order to get some intuition about the general structure of Hodge-type cohomological theories it would be surely very useful to find other examples in higher dimensions.

We have conjectured that the higher dimensional analogue of the Blau–Thompson model is the only topological one which can be constructed in  $D = 5$ . In order to prove that, it is necessary to go into a closer analysis of the structure of all the other cohomological gauge theories in  $D = 5$ .

**Acknowledgement** One of the authors (B.G.) kindly acknowledges the warm hospitality of the Institute of Physics, Sao Paulo University, where this work was finished, and the financial support of this stay by DAAD – FAPESP scientific exchange program. He also expresses his sincere thanks to D. Gitman for various enlightening discussions.

## References

- [1] for a review, see, J. W. van Holten, *Aspects of BRST quantization*, Lectures at Summer School *Geometry and Topology in Physics*, Rot a.d. Rot (Germany), Sept. 2001 (hep-th/0201124), and references therein
- [2] B. Geyer and D. Mülsch, *Phys. Lett. B* **518** (2001) 181; *Hodge-type cohomological gauge theories*, in: Proc. Fifth Int. Alexander Friedman Seminar, Joao Pessoa, April 24 - 30, 2002, *Int. J. Mod. Phys. A* **17** (2002) 4425
- [3] M. Blau and G. Thompson, *Nucl. Phys. B* **492** (1997) 545
- [4] J. W. van Holten, *Phys. Rev. Lett.* **64** (1990) 2863
- [5] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, *Phys. Lett. B* **62** (1976) 105; P. Gepner, *Nucl. Phys. B* **296** (1988) 757; W. Lerche, C. Vafa and N.P. Warner, *Nucl. Phys. B* **324** (1989) 427; B. Green, C. Vafa and N.P. Warner, *Nucl. Phys. B* **324** (1989) 371
- [6] P. Candelas, C.T. Horowitz, A. Strominger and E. Witten, *Nucl. Phys. B* **258** (1985) 46; P. Candelas, A.M. Dale, C.A. Lutken and R. Schimmrigk, *Nucl. Phys. B* **298** (1988) 493; P. Candelas, *Nucl. Phys. B* **298** (1989) 458; P. Candelas and X. de la Ossa, *Nucl. Phys. B* **342** (1990) 246
- [7] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys. B* **352** (1991) 59; E. Verlinde and N.P. Warner, *Phys. Lett. B* **269** (1991) 96
- [8] L. Baulieu, H. Kanno and I. Singer, *Commun. Math. Phys.* **194** (1998) 149; *Cohomological Yang-Mills theory in eight dimensions*, in: *Dualities in String Theories*, Seoul 1997, p.365, hep-th/9705127
- [9] B.S. Acharya and M. O'Loughlin, *Phys. Rev. D* **55** (1997) 4521, B. S. Acharya, M. O'Loughlin and B. Spence, *Nucl. Phys. B* **503** (1997) 657
- [10] M. Blau and G. Thompson, *Phys. Lett. B* **415** (1997) 242
- [11] N. Marcus, *Nucl. Phys. B* **452** (1995) 331;
- [12] B. Geyer and D. Mülsch, *Nucl. Phys. B* **616** (2001) 476
- [13] B. Geyer and D. Mülsch,  $N_T = 8$ ,  $D = 2$  *Hodge-type cohomological gauge theory with global  $SU(4)$  symmetry*, Proc. 3. Int. Andrei Sakharov Conference on Physics, Moscow, June 24 - 29, 2002; hep-th/0210268
- [14] E. Witten, *Nucl. Phys. B* **323** (1989) 113
- [15] M. Bershadsky, V. Sadov and C. Vafa, *Nucl. Phys. B* **448** (1995) 166
- [16] K. Nishijima, *Prog. Theor. Phys.* **80** (1988) 897; **80** (1988) 905
- [17] A.V. Razumov and G.N. Rybkin *Nucl. Phys. B* **332** (1990) 209
- [18] L. Brink, J. Schwarz and J. Scherk, *Nucl. Phys. B* **121** (1977) 77
- [19] C.Vafa and E. Witten, *Nucl. Phys. B* **431** (1994) 3; M. Blau and G. Thompson, *Commun. Math. Phys.* **152** (1993) 41; R. Dijkgraaf and G. Moore, *Commun. Math. Phys.* **185** (1997) 411

- [20] J. Yamron, *Phys. Lett.* **B 213** (1988) 325
- [21] W. Siegel and M. Rocek, *Phys. Lett.* **B 105** (1981) 275; V. Rivelles and J.G. Taylor, *J. Phys. A: Math. Gen.* **A 15** (1982) 163
- [22] P. van Baal, S. Ouvry and R. Stora, *Phys. Lett.* **B 220** (1989) 159; F. Delduc, N. Maggiore, O. Piguet and S. Wolf, *Phys. Lett.* **B 385** (1996) 132